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Original research article

Control charts for processes with variable mean

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Abstract: Despite of strong ability and performance of control charts to control and monitor processes, they have some problems in practical applications. If control chart's limits are not properly designed then we receive false alarms. For example, several observations may be outside the control limits when the mean of process is in-control. Not considering the variation of the process mean at each sampling time may lead to this error. The process may be adjusted at specific mean but different working conditions and different operators may change mean of the process and it may have a small deviation from its predetermined value and this problem can lead to wrong implementation of control charts. In this paper, the effects of variable mean on control charts are analyzed. It is assumed that the mean of observation varies over time but its probability distribution is normal probability distribution function. It is observed that long-term process mean control chart generates false alarms.

Keywords: variable mean; Shewhart control chart; control limits; first type error

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1. Introduction

In the processes that are monitored using control charts, there are always random deviations from target value which are divided into two types: deviations with special causes and random deviations. Random deviations normally exist in the process so that the performance of the process is acceptable and the process is in-control in spite of these deviations. But in the case of deviations due to the special causes, the performance of the process is not acceptable and the process would be out of control. The process is controlled by control charts to reduce these deviations and if there is unacceptable performance, the process is inspected and adjusted due to special causes of variations and possible failure are repaired[1].

Control limits in control charts are designed so that they can reduce probability of first type error (the probability of rejecting in-control process) until it reduces to an acceptable value. Determining this error is dependent on economic factors. Probability of first type error is usually assumed to be 0.0027 for observations that are normally distributed and control limits are designed with this value[1].

Determining control limits chart is one of the most important factors in quality control, because wrong design of limits, leads to produce defectives. Also, additional cost is incurred for the inspecting and identifying special causes of deviations while the process is in-control and the wrong design of limits results in wrong decision of out-of-control process.

Studies undertaken in field of control charts are extensive. Shewhart[2] presented principles of control charts. Ryan[3] presented control limits for S charts. Brown[4] used binary characterized function for classifications of observations in Shewhart control charts, also Senturk and Erginel[5] conducted researches in preparation of Shewhart charts and fuzzy control charts and other researchers like Lee[6] presented performance evaluation of adaptive control charts by Markov models, Torang et al.[7] presented optimization model to design sample size, control warnings limits, $\bar{X}$ and R control limits and sampling interval by adding constraints to the cost model of Duncan, Schilling and Nelson[8] presented control charts for non-normal data, Fallah Nezhad and Akhavan Niaki[9] presented a method for analyzing and classifying univariate quality control systems using a recursive method and in other studies carried out in the field of Shewhart control charts, mean process is assumed to be a constant[1].

Malindzakova et al.[10] examined the implementation of Shewhart control charts in quality and production management. With the help of these charts, quality control and production process control can be easily carried out. The article explains how to implement Shewhart control charts for production and quality management.

Nagar[11] defines statistical quality control and discusses various methods of using it. Additionally, the article examines how statistical quality control is used in industrial production and presents methods for improving quality. Finally, the author discusses the advantages and disadvantages of using statistical quality control and presents their conclusion.

Triantafyllou and Ram[12] examined distribution-free CUSUM control charts for monitoring industrial processes. The article explains how to use distribution-free CUSUM control charts for monitoring industrial processes. The authors are looking for ways to improve quality and efficiency in industry, and they use distribution-free CUSUM control charts as their main tool.

Brown[13] considered monitoring a multiple stream process (MSP) where multiple processes which are assumed to be identical are being simultaneously monitored, but the assumption of normality cannot be reasonably be verified, the NEMT-CUSUM control chart has been proposed as a possible non parametric alternative. The purpose of this study was to derive the alternative distribution of the NEMT-CUSUM plotting statistic such that statistical power can be estimated.

In some processes, the mean value may not be constant and its value changes at each time of sampling, but it will fluctuate around a constant value. When the operator adjusts the machine, he usually adjusts the machine on particular target values but these settings are not fixed and has a small deviation around the target value according to the working conditions, different operators' skills that are working on the machine or the vibration of machine or temperature that these deviations can effect on the control chart.

In fact, it is possible that the deviations of the observations within the sample are not considered in the design of control limits. Assume that the machine is adjusted on process mean μ_i at *i*-th time of sampling where μ_i follows a normal distribution with parameters (μ_0, σ_0^2) where μ_0 is the target value and σ_0^2 is the variance. Thus, the observations are normally distributed with parameters (μ_i, σ_1^2) at *i*-th time of sampling. The objective of this research is to analyze the mentioned data in control charts. First, we show that control charts classify these data as out of control[1]. However it is the contention of this study that mean value of observation in such process remains constant thus the process is actually in control and the result of applying control chart is wrong and it is proven that the problem in hand is due to under estimation of the standard deviation in traditional control charts.

2. Problem statement

Consider a situation where a machine is set to a specific adjustment by an operator every day. It is clear that it is expected to have some deviation in the adjustment of different operators in different days. This problem leads us to the issue that the mean value of process may not be a constant number and may follow a normal distribution. In this article, we show that in this situation, the performance of standard control charts is disturbed and they may produce incorrect alarms due to the fact that the estimate of the standard deviation in the control charts will be incorrect. In this study, we have developed a solution to such a problem and developed methods for the proper performance of the control charts.

Even in automatic production systems, when equipment is in different environmental and temperature conditions, default equipment settings may deviate slightly from the standard values. Ignoring this deviation will lead to disruption of the quality control process for the produced parts.

The main contribution of this article is that methods for accurately estimating the standard deviation and coefficient of standard deviation within the control chart limits have been presented. These methods provide a solution for the problem of quality control of a process with a variable mean. By "variable mean", we mean that the mean of a random variable is itself a random variable that follows a specified probability distribution, which in this case we have assumed to be a normal distribution. In fact, the main idea of the article is to understand the behavior of the processes with a variable mean, which in practical situations leads to a disturbance in the performance of standard control charts.

In what follows, first we simply illustrate the problem in the performance of decision-making of control charts when the mean parameter of process is randomized and then we develop the solution of modifying the calculation of control limits.

3. Analyzing the effect of variable mean on performance of the control chart

The simulation data is obtained using MATLAB software so that first, we generated 20,000 random mean values μ_i from a normal distribution with mean μ_0 and variance σ_0^2 and then 5 observation are generated with mean μ_i (the μ_i is the mean obtained from normal distribution (μ_0, σ_0^2) and variance σ_1^2 at i-th time of sampling. We simulated 11 different scenarios of parameters to enhance the accuracy of the simulation study, where, 1,000,000 observations are generated for each one.

We actually generated 20,000 samples with samples size of 5 for each scenario and they have been analyzed using Minitab software and probability of first type error is given in the last column of **Table 1** for each scenario.

It can be concluded from **Table 1** that the probability of first type error is large with regards to standard error (α = 0.0027) in the most examples of generated data, in the other words many data are placed outside of the control limits. For analyzing this issue, we must examine the type of data distribution in order to determine the reason for wrong performance of control limits. As described in the steps of simulation. First mean value of observations is generated using normal distribution in Equation (1), then these mean values are used for generating the observations at each sampling point using normal distribution in Equation (2). Since our observations are generated using conditional distribution in Equation (2), thus the observations are generated with mean values μ_i where μ_i follow normal distribution, thus the concept of conditional distribution is employed here and we obtain distribution of the observations using formulations of conditional distribution in Equation (4).

Table 1. Performance of control chart for process with variable mean.

Scenario	$\boldsymbol{\mu_i \sim N(\mu_0, \sigma_0^2)}$	$\boldsymbol{\bar{X} \sim N\left(\mu_i, \frac{\sigma_1^2}{n}\right)}$	$\boldsymbol{C = Z_{\frac{\alpha}{2}}}$	$\boldsymbol{\alpha}$
1	$\mu_1 \sim N(0; 0.5)$	$X \sim N(\mu_1; 2)$	3	0.00895
2	$\mu_2 \sim N(0; 1.5)$	$X \sim N(\mu_2; 2.4)$	3	0.08465
3	$\mu_3 \sim N(0; 0.005)$	$X \sim N(\mu_3; 1.005)$	3	0.00235
4	$\mu_4 \sim N(0; 0.3)$	$X \sim N(\mu_4; 3)$	3	0.0042
5	$\mu_5 \sim N(0; 1)$	$X \sim N(\mu_5; 1)$	3	0.22035
6	$\mu_6 \sim N(0; 0.03)$	$X \sim N(\mu_6; 1.04)$	3	0.0026
7	$\mu_7 \sim N(0; 1.3)$	$X \sim N(\mu_7; 3.2)$	3	0.02535
8	$\mu_8 \sim N(0; 0.01)$	$X \sim N(\mu_8; 1.01)$	3	0.0026
9	$\mu_9 \sim N(0; 0.8)$	$X \sim N(\mu_9; 2.8)$	3	0.0149
10	$\mu_{10} \sim N(0; 0.015)$	$X \sim N(\mu_{10}; 1.9)$	3	0.00265
11	$\mu_{11} \sim N(0; 0)$	$X \sim N(0; 1)$	3	0.0026

4. The marginal distribution of *X*

It is clear that conditional distribution $x \mid \mu$ and mean of observations at each time of sampling follow the following distributions:

$$f(\mu) = N(\mu_0, \sigma_0^2) \tag{1}$$

$$f(X \mid \mu) = N(\mu, \sigma_1^2) \tag{2}$$

So, using normal probability density, the marginal probability distribution of *X* is calculated as follows:

$$f(\mu) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}}$$

$$f(x \mid \mu) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma_1^2}} \tag{3}$$

$$f(x) = \int f(x \mid \mu) f(\mu) d\mu \tag{4}$$

Thus using conditional distribution formula, following is obtained,

$$f(x) = \int f(x \mid \mu) f(\mu) d\mu = \int \frac{1}{2\pi\sigma_0\sigma_1} e^{-\frac{(x-\mu)^2}{2\sigma_0^2}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_1^2}} d\mu = \frac{\sqrt{\sigma_0^2+\sigma_1^2}}{\sqrt{\sigma_0^2+\sigma_1^2}} \times \int \frac{1}{2\pi\sigma_0\sigma_1} e^{-\frac{1}{2}\left[\left(\frac{x-\mu}{\sigma_1}\right)^2 + \left(\frac{\mu-\mu_0}{\sigma_0}\right)^2\right]} d\mu$$
$$\int \frac{\sqrt{\sigma_0^2+\sigma_1^2}}{\sqrt{2\pi}\sigma_0\sigma_1} e^{-\frac{1}{2}\frac{(\sigma_0^2+\sigma_1^2)(\mu-\mu')^2}{\sigma_0^2\sigma_1^2}} d\mu \times \frac{1}{\sqrt{\sigma_0^2+\sigma_1^2} \times \sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu_0)^2}{(\sigma_0^2+\sigma_1^2)}} \tag{5}$$

where,

$$\mu' = \frac{\sigma_0^2 x + \sigma_1^2 \mu_0}{\sigma_0^2 + \sigma_1^2} \tag{6}$$

Since $\int \frac{\sqrt{\sigma_0^2+\sigma_1^2}}{\sqrt{2\pi}\sigma_0\sigma_1} e^{-\frac{1}{2}\frac{(\sigma_0^2+\sigma_1^2)(\mu-\mu')^2}{\sigma_0^2\sigma_1^2}} d\mu = 1$, thus following is obtained,

$$f(x) = \frac{1}{\sqrt{\sigma_0^2+\sigma_1^2} \times \sqrt{2\pi}} e^{-\frac{1(x-\mu_0)^2}{2(\sigma_0^2+\sigma_1^2)}} \tag{7}$$

What is obtained from the above process $f(x)$is the normal distribution. Thus, the mean value of observations is actually equal μ_0 and their variance is $\sigma_0^2 + \sigma_1^2$ where σ_0 the standard deviation of mean values is and σ_1 is the standard deviation of the conditional distribution of observations.

However, since our observations are normally distributed with constant mean μ_0 thus their performance in Shewhart control charts, should lead to acceptable results and probability of first type error should be close to 0.0027. However, according to what was observed in **Table 1**; the performance of control charts is not acceptable. Thus control limits of Shewhart control chart are not working properly for these observations so control limits should be designed so that the probability of first type error in such observations closes to 0.0027. It is known that the estimator $\frac{\bar{R}}{d_2}$ would be used for estimating the standard deviation but this estimator in explained problem estimates the value of σ_1 thus ignoring the value of σ_0 in designing the control limits leads to false alarm because of down estimation of standard deviation of the process thus first we try to estimate the correct value of standard deviation of $\bar{X}$ and then this estimator would be used for designing the control limits. The method of solving this problem is presented in the following.

If the observations follow normal distribution with mean μ and standard deviation σ_1 and mean value (μ) was constant then the average of data in samples follow normal distribution with mean (μ) and standard deviation $\sigma_{\bar{X}} = \sqrt{\frac{\sigma_1^2}{n}}$ but mean value of our observations in each sampling point is not fixed and has variations around a target value thus we should consider the distribution of mean values.

Generated observations follow a conditional normal distribution with mean μ_i and variance σ_1^2 and the mean values (μ_i) are normally distributed with mean μ_0 and variance σ_0^2 . Thus, the marginal distribution of $\bar{X}$ is determined as follows:

$$f(X \mid \mu) = N(\mu, \sigma_1^2) \Rightarrow f(\bar{X} \mid \mu) = N\left(\mu, \frac{\sigma_1^2}{n}\right) = \frac{1}{\sqrt{\frac{\sigma_1^2}{n}}\sqrt{2\pi}} \quad e^{-\frac{(\bar{X}-\mu)^2}{2\frac{\sigma_1^2}{n}}} \tag{8}$$

Thus using conditional distribution formula, following is obtained,

$$f(\bar{X}) = \int f(\bar{X} \mid \mu) f(\mu) \mathrm{d}\mu = \int \frac{1}{2\pi\sigma_0 \frac{\sigma_1}{\sqrt{n}}} e^{-\frac{(\bar{X}-\mu)^2}{2\frac{\sigma_1^2}{n}}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}} d\mu = \frac{\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{n}}}{\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{n}}} \times \int \frac{1}{2\pi\sigma_0 \frac{\sigma_1}{\sqrt{n}}} e^{-\frac{1}{2}\left[\left(\frac{\bar{X}-\mu}{\frac{\sigma_1}{n}}\right)^2 + \left(\frac{\mu-\mu_0}{\sigma_0}\right)^2\right]} d\mu =$$
$$\int \frac{\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{n}}}{\sqrt{2\pi}\sigma_0 \frac{\sigma_1}{\sqrt{n}}} e^{-\frac{1}{2}\frac{\left(\sigma_0^2 + \frac{\sigma_1^2}{n}\right)(\mu - c')^2}{\sigma_0^2 \frac{\sigma_1^2}{n}}} d\mu \frac{1}{\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{n}} \times \sqrt{2\pi}} e^{-\frac{1}{2}\frac{(\bar{X}-\mu_0)^2}{(\sigma_0^2 + \frac{\sigma_1^2}{n})}} \tag{9}$$

$$c' = \frac{\sigma_0^2 \bar{X} + \frac{\sigma_1^2}{n}\mu_0}{\sigma_0^2 + \frac{\sigma_1^2}{n}}$$

Since $\int \frac{\sqrt{\sigma_0^2+\frac{\sigma_1^2}{n}}}{\sqrt{2\pi}\sigma_0\frac{\sigma_1}{\sqrt{n}}} e^{-\frac{1}{2}\frac{\left(\sigma^2+\frac{\sigma_1^2}{n}\right)(\mu-c\prime)^2}{\sigma_0^2\frac{\sigma_1^2}{n}}} d\mu = 1$, thus following is obtained,

$$f(\bar{X}) = \frac{1}{\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{n}} \times \sqrt{2\pi}} e^{-\frac{1}{2}\frac{(\bar{X}-\mu_0)^2}{(\sigma_0^2+\frac{\sigma_1^2}{n})}}$$

Thus, it is obtained that $\bar{X}$ follows normal distribution with mean μ_0 and its variance is $\sigma_0^2 + \frac{\sigma_1^2}{n}$. This denotes an important role for the standard deviation of the mean value σ_0 in calculating the standard deviation of $\bar{X}$. Therefore, if observations are obtained from a process with variable mean, then instead of using $\sigma_{\bar{X}} = \sqrt{\frac{\sigma_1^2}{n}}$ for determining control limits, we should apply the formula $\sigma_{\bar{X}} = \sqrt{\sigma_0^2 + \frac{\sigma_1^2}{n}}$ to obtain the limits. It is denoted that such revision of control limits results in acceptable value for the probability of first type errors.

5. Method of estimating the parameters $\sigma_0$and σ_1

To compute new limits control, we need to estimate the parameters σ_0 and σ_1. The values of σ_0 and σ_1 can be estimated from following equation:

$$\hat{\sigma}_1 = \frac{\bar{R}}{d_2}, \quad \bar{R} = \frac{\sum_i (x_{i,\max} - x_{i,\min})}{m-1} \tag{10}$$

$$\hat{\sigma}_0 = \frac{\overline{MR}}{1.128}, \quad \overline{MR} = \frac{\sum_i |\bar{x}_i - \bar{x}_{i-1}|}{m-1} \tag{11}$$

The Equation (10) is used to estimate the standard deviation of each observation without considering the variations between the samples. This provides an estimate of the standard deviation of the observations in the process that is the standard deviation within the samples. Additionally, Equation (11) is used to estimate the standard deviation of the process mean based on the variations between sample means. This provides an estimate of the standard deviation between the samples, which is equivalent to the standard deviation of the process mean.

Therefore, the variance of $\bar{X}$ can be obtained from following equation:

$$\sigma_{\bar{X}}^2 = \hat{\sigma}_0^2 + \frac{\hat{\sigma}_1^2}{n}$$

The results of estimating the parameters are denoted in **Table 2**.

As denoted in **Table 2**, the estimates of standards deviations are close to their exact values used to generate initial data. $\hat{\sigma}_0$ is the estimation of σ_0 and $\hat{\sigma}_1$ is the estimation of σ_1. The results show accuracy of equations for the estimation of the standard deviation.

Also, accuracy of the equation $\sigma_{\bar{X}} = \sqrt{\sigma_0^2 + \frac{\sigma_1^2}{n}}$ in the processes with variable mean is tested in control charts. The results have been denoted in **Table 3**.

Table 2. Comparing estimations of σ_0 and σ_1 with their values in simulation.

Scenarios	σ_0	$\hat{\sigma}_0$	σ_1	$\hat{\sigma}_1$
1	0.5	0.517	2	1.997
2	1.5	1.517	2.4	2.4008
3	0.005	0.00493	1.005	1.008
4	0.3	0.3	3	2.996
5	1	0.985	1	1.0014
6	0.03	0.0284	1.04	1.0415
7	1.3	1.27	3.2	3.197
8	0.01	0.018	1.01	1.011
9	0.8	0.791	2.8	2.808
10	0.015	0.0148	1.9	1.902
11	0	0.001	1	1.001

Table 3. The results of using estimated standard deviation in the process with variable mean.

Scenarios	$\mu_i \sim N(\mu_0, \hat{\sigma}_0^2)$	$X \sim N(\mu_i, \hat{\sigma}_1^2)$	$\hat{\sigma}_{\bar{X}}=\sqrt{\sigma_0^2+\frac{\sigma_1^2}{5}}$	c	UCL	α
1	$\mu_1 \sim N(0; 0.5170)$	$X \sim N(\mu_1; 1.997)$	1.032	3	3.096	0.00255
2	$\mu_2 \sim N(0; 0.517)$	$X \sim N(\mu_2; 2.400)$	1.859	3	5.577	0.00265
3	$\mu_3 \sim N(0; 0.00493)$	$X \sim N(\mu_3; 1.008)$	0.451	3	1.353	0.0026
4	$\mu_4 \sim N(0; 0.3)$	$X \sim N(\mu_4; 2.996)$	1.396	3	4.189	0.00285
5	$\mu_5 \sim N(0; 0.985)$	$X \sim N(\mu_5; 1.001)$	1.082	3	3.246	0.00285
6	$\mu_6 \sim N(0; 0.0284)$	$X \sim N(\mu_6; 1.0415)$	0.466	3	1.399	0.00255
7	$\mu_7 \sim N(0; 1.27)$	$X \sim N(\mu_7; 3.197)$	1.915	3	5.746	0.00285
8	$\mu_8 \sim N(0; 0.018)$	$X \sim N(\mu_8; 1.011)$	0.452	3	1.358	0.0026
9	$\mu_9 \sim N(0; 0.791)$	$X \sim N(\mu_9; 2.8085)$	1.484	3	4.453	0.0026
10	$\mu_{10} \sim N(0; 0.0148)$	$X \sim N(\mu_{10}; 1.902)$	0.851	3	2.553	0.0026
11	$\mu_{11} \sim N(0; 0.001)$	$X \sim N(\mu_{11}; 0.001)$	0.447	3	1.343	0.0026

The results show that the probability of first type error in the control chart with new limits is satisfactory and acceptable and it is close to 0.0027. To analyze the role of the coefficient of standard deviation, this parameter is adjusted in order to obtain the exact value of $\alpha = 0.0027$ for the probability of first type error of control chart and its value is denoted with parameter c''. The results are denoted in **Table 4**.

We do an analysis to determine whether the coefficient of standard deviation is correlated with the values of σ_0 and σ_1 by using “Eviews” software. This can be performed using the hypothesis testing for coefficient of regression function.

Thus, we adjusted the value of c in each scenario so that the probability of first type error exactly equals 0.0027. Now the assumption that the coefficient of standard deviation has correlation with standard deviation is tested and we want to find the relationship between σ_0 and σ_1. We obtained a regression function for c'' based on the values of σ_0 and σ_1 using “Eviews” software.

Table 4. Acceptable performance of control charts for observations of variable mean.

Scenarios	$\mu_i \sim N(\mu_0, \sigma_0^2)$	$X \sim N(\mu_i, \sigma_1^2)$	$\hat{\sigma}_{\bar{X}}=\sqrt{\sigma_0^2+\frac{\sigma_1^2}{5}}$	c''	*UCL*	α
1	$\mu_1 \sim N(0; 0.5170)$	$X \sim N(\mu_1; 1.997)$	1.032	2.966	3.061	0.0027
2	$\mu_2 \sim N(0; 1.517)$	$X \sim N(\mu_2; 2.400)$	1.859	2.982	5.545	0.0027
3	$\mu_3 \sim N(0; 0.00493)$	$X \sim N(\mu_3; 1.008)$	0.451	2.962	1.336	0.0027
4	$\mu_4 \sim N(0; 0.3)$	$X \sim N(\mu_4; 2.996)$	1.396	3.058	1.393	0.0027
5	$\mu_5 \sim N(0; 0.985)$	$X \sim N(\mu_5; 1.001)$	1.082	3.03	2.28	0.0027
6	$\mu_6 \sim N(0; 0.0284)$	$X \sim N(\mu_6; 1.0415)$	0.466	2.985	4.205	0.0027
7	$\mu_7 \sim N(0; 1.27)$	$X \sim N(\mu_7; 3.197)$	1.915	3.031	5.806	0.0027
8	$\mu_8 \sim N(0; 0.018)$	$X \sim N(\mu_8; 1.011)$	0.452	2.986	1.351	0.0027
9	$\mu_9 \sim N(0; 0.791)$	$X \sim N(\mu_9; 2.8085)$	1.484	2.986	4.432	0.0027
10	$\mu_{10} \sim N(0; 0.0148)$	$X \sim N(\mu_{10}; 1.902)$	0.851	2.986	2.541	0.0027
11	$\mu_{11} \sim N(0; 0.001)$	$X \sim N(0; 1.001)$	0.447	3.123	1.399	0.0027

6. Estimation of regression function

Result of regression obtained for coefficient of standard deviation in control chart is as follows:

$$c'' = 3.005899 - 0.00905\sigma_0 + 0.0039\sigma_1 \tag{12}$$

The coefficient of standard deviation in control limits is estimated using linear regression based on the standard deviation of the process mean and the standard deviation within the samples in Equation (12) to ensure that the type I error rate will be equal to the standard value. The coefficient obtained from this equation can be used to calculate the adjusted upper and lower control limits.

Other results and outputs of estimation obtained by "Eviews" are shown in **Table 5**. As can be seen we have examined 1) the regression function, 2) the regression coefficients and 3) the linearity of regression such that the error terms would be uncorrelated and to have heterogeneous variance.

The general form of the regression function is as follows.

$$c'' = \beta_0\sigma_0 + \beta_1\sigma_1 + \beta \tag{13}$$

For examining the regression function, we used following hypothesis to check whether there is any linear relation among parameters,

$$\begin{gathered} H_0: \beta_0 = \beta_1 = 0 \\ H_1: \text{Otherwise} \end{gathered} \tag{14}$$

As shown in **Table 5**, the F-statistic is equal to 0.03 and it is smaller than $F_{2,8,0.01} = 8.65$ thus the hypothesis H_0 is accepted. Thus, hypothesis one is not accepted with probability of 99%, resulting that regression function is not suitable.

For examining the regression coefficient, we used following hypothesis.

$$\begin{gathered} H_0: \beta_0 = 0 \\ H_1: \beta_0 \neq 0 \end{gathered} \tag{15}$$

Table 5. Estimated values for coefficients of standard deviation in control chart.

Dependent variable: c''				
Method: least squares			-	-
Included observations: 11			-	-
Variable	Coefficient	Std. error	T-statistic	Prob.
C	3.005899	0.039462	76.17235	0.0000
σ_0	−0.009050	0.036651	−0.246924	0.8112
σ_1	0.003900	0.023251	0.167719	0.8710
R-squared	0.007701	Mean dependent var		3.008636
Adjusted R-squared	−0.240374	S.D. dependent var		0.048240
S.E. of regression	0.053725	Akaike info criterion		−2.782862
Sum squared residual	0.023091	Schwarz criterion		−2.674345
Log likelihood	18.30574	Hannan-Quinn criterion		−2.851266
F-statistic	0.031044	Durbin-Watson stat		1.606623
Prob (F-statistic)	0.969549	-	-	-

As shown in **Table 5** the T-statistic is equal to -0.246 and its absolute value is smaller than $t_{8,0.01}$ =3.355, thus the null hypothesis is accepted. Thus, hypothesis one is not accepted with probability of 99%, resulting that the regression coefficient β_0 is equal zero.

Following hypothesis is examined to check the value of β_1.

$$H_0: \beta_1 = 0$$
$$H_1: \beta_1 \neq 0 \tag{16}$$

As shown in **Table 5**, the T-statistic is equal to 0.197 and it is smaller than $t_{8,0.01}$ =3.355, thus the null hypothesis is accepted. Thus, hypothesis one is not accepted with probability of 99%, resulting that the regression coefficient β_1 is equal zero.

After examining parameters of the regression and the regression coefficients, we conclude that there is not any correlation among them thus we can say that the numerical value of c'' is close to 3 now once again we check the **Table 3** to review the probabilities of the first type error resulted from the value $C = 3$ for coefficient of standard deviation. Now the standard error of these probabilities with regards to their standard value $\left(\frac{|\alpha_i - 0.0027|}{0.0027}\right)$ are evaluated. The results are expressed in **Table 6**.

As shown in **Table 6**, the values of standard error $\frac{|\alpha_i - 0.0027|}{0.0027}$ in analyzed examples are about equal to 0.05 that is denoting a satisfactory result for choosing $c = 3$ as the coefficient of standard deviation in control limits.

After assessments carried out on the parameters of control limits, we realized that the data with variables mean are actually normally distributed with constant mean thus the standard deviation of observations are obtained using conditional probability formula and we achieved the following equation as correct control limits after considering coefficient of standard deviation.

$$UCL = \mu + 3\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{n}}$$
$$CL = \mu \qquad (17)$$
$$LCL = \mu - 3\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{n}}$$

Table 6. Evaluation of standard error for probabilities of the type first error in control chart.

Scenarios	$\mu_i \sim N(\mu_0, \hat{\sigma}_0^2)$	$X \sim N(\mu_i, \hat{\sigma}_1^2)$	$\hat{\sigma}_{\bar{X}} = \sqrt{\sigma_0^2 + \frac{\sigma_1^2}{5}}$	c	α	$\frac{\lvert \alpha_i - 0.0027 \rvert}{0.0027}$
1	$\mu_1 \sim N(0; 0.5170)$	$X \sim N(\mu_1; 1.997)$	1.032	3	0.00255	0.055556
2	$\mu_2 \sim N(0; 1.517)$	$X \sim N(\mu_2; 2.400)$	1.859	3	0.00265	0.018519
3	$\mu_3 \sim N(0; 0.00493)$	$X \sim N(\mu_3; 1.008)$	0.451	3	0.0026	0.037037
4	$\mu_4 \sim N(0; 0.3)$	$X \sim N(\mu_4; 2.996)$	1.396	3	0.00285	0.055556
5	$\mu_5 \sim N(0; 0.985)$	$X \sim N(\mu_5; 1.001)$	1.082	3	0.00285	0.055556
6	$\mu_6 \sim N(0; 0.0284)$	$X \sim N(\mu_6; 1.0415)$	0.466	3	0.00255	0.055556
7	$\mu_7 \sim N(0; 1.27)$	$X \sim N(\mu_7; 3.197)$	1.915	3	0.00285	0.055556
8	$\mu_8 \sim N(0; 0.018)$	$X \sim N(\mu_8; 1.011)$	0.452	3	0.0026	0.037037
9	$\mu_9 \sim N(0; 0.791)$	$X \sim N(\mu_9; 2.8085)$	1.484	3	0.0026	0.037037
10	$\mu_{10} \sim N(0; 0.0148)$	$X \sim N(\mu_{10}; 1.902)$	0.851	3	0.0026	0.037037
11	$\mu_{11} \sim N(0; 0.001)$	$X \sim N(\mu_{11}; 1)$	0.447	3	0.0026	0.037037

7. Effect of variable mean on the performance of R control chart

As mentioned in previous section, 11 examples with the variable mean are simulated where each one includes 20,000 samples with sample size of 5 and using Minitab software the control limits are determined for control charts. The probability of first type error has been calculated using obtained control limits. The results as well as the number of points that are outside the control limits have been reported in **Table 7**.

Table 7. Results of the data with variables mean in R control chart.

-	$\mu_i \sim N(\mu_0, \hat{\sigma}_0^2)$	$X \sim N(\mu_i, \hat{\sigma}_1^2)$	UCL	α	Number of points outside the control limits
1	$\mu_1 \sim N(0; 0.5170)$	$X \sim N(\mu_1; 1.997)$	9.8	0.00415	83
2	$\mu_2 \sim N(0; 1.517)$	$X \sim N(\mu_2; 2.400)$	11.9	0.00415	83
3	$\mu_3 \sim N(0; 0.00493)$	$X \sim N(\mu_3; 1.008)$	4.9	0.00425	85
4	$\mu_4 \sim N(0; 0.3)$	$X \sim N(\mu_4; 2.996)$	14.74	0.00425	85
5	$\mu_5 \sim N(0; 0.985)$	$X \sim N(\mu_5; 1.001)$	4.9	0.00415	83
6	$\mu_6 \sim N(0; 0.0284)$	$X \sim N(\mu_6; 1.0415)$	5.1	0.0042	84
7	$\mu_7 \sim N(0; 1.27)$	$X \sim N(\mu_7; 3.197)$	15.7	0.00425	85
8	$\mu_8 \sim N(0; 0.018)$	$X \sim N(\mu_8; 1.011)$	4.94	0.00425	85
9	$\mu_9 \sim N(0; 0.791)$	$X \sim N(\mu_9; 2.8085)$	13.8	0.00425	85
10	$\mu_{10} \sim N(0; 0.0148)$	$X \sim N(\mu_{10}; 1.902)$	9.3	0.0042	84
11	$\mu_{11} \sim N(0; 0.001)$	$X \sim N(\mu_{11}; 1)$	4.9	0.00415	83

We examined the probability of first type error of different cases. As seen in the **Table 7**, all of the probabilities of first type error are sufficiently close to the standard error that is reported in case 11 (it is assumed that the value of σ_0 is negligible which is the case of standard normal observations). This indicates that the data with variable mean had no effect on the R control charts and acceptable results are obtained with the control limits of this chart. Although if we use $c = 3$ in control limits of $\bar{X}$ control chart in case of normal distribution for quality characteristics then the probability of first type error will be equal to 0.0027 but this result is not true in the case of R control charts. Distribution of range of samples even when the distribution of data is normal is not symmetric. In fact, the shape of the distribution has skewness to the right and the probability of first type error is equal to 0.00415 for samples with size of $n = 5$.

8. Effect of variable mean on the performance of S control chart

Similar analysis is performed for S control chart using Minitab statistical software to calculate control limits and the probability of first type error as well as the number of samples that are outside the control limits. The results have been reported in **Table 8**.

Table 8. Results of the data with variables mean in S control chart.

	$\boldsymbol{\mu_i \sim N(\mu_0, \hat{\sigma}_0^2)}$	$\boldsymbol{X \sim N(\mu_i, \hat{\sigma}_1^2)}$	***UCL***	$\boldsymbol{\alpha}$	**number of points outside the control limits**
1	$\mu_1 \sim N(0; 0.5170)$	$X \sim N(\mu_1; 1.997)$	3.93	0.0036	71
2	$\mu_2 \sim N(0; 1.517)$	$X \sim N(\mu_2; 2.400)$	4.7	0.00355	71
3	$\mu_3 \sim N(0; 0.00493)$	$X \sim N(\mu_3; 1.008)$	1.98	0.0038	76
4	$\mu_4 \sim N(0; 0.3)$	$X \sim N(\mu_4; 2.996)$	5.86	0.00365	73
5	$\mu_5 \sim N(0; 0.985)$	$X \sim N(\mu_5; 1.001)$	1.96	0.00355	71
6	$\mu_6 \sim N(0; 0.0284)$	$X \sim N(\mu_6; 1.0415)$	2.044	0.00355	71
7	$\mu_7 \sim N(0; 1.27)$	$X \sim N(\mu_7; 3.197)$	6.3	0.0036	72
8	$\mu_8 \sim N(0; 0.018)$	$X \sim N(\mu_8; 1.011)$	1.985	0.00355	71
9	$\mu_9 \sim N(0; 0.791)$	$X \sim N(\mu_9; 2.8085)$	5.53	0.0036	72
10	$\mu_{10} \sim N(0; 0.0148)$	$X \sim N(\mu_{10}; 1.902)$	3.73	0.00355	71
11	$\mu_{11} \sim N(0; 0.001)$	$X \sim N(\mu_{11}; 1)$	1.96	0.00355	71

Since the results of case 11 in **Table 8** come from standard normal distribution thus we compare other cases with this case. As seen in the **Table 8**, the values of the probability of first type error in this chart for different scenarios are also very close to the standard value of the probability of first type error in standard normal case thus the variable mean has not effected on the performance of S chart. Also, it is seen that the probability of first type error is 0.00355 for S control chart with the sample size of $n = 5$.

9. Effect of variable mean on the performance of control chart for individual observations

We used simulation to analyze the effect of such data on control chart for individual observation. First, we generated mean values μ_i from the normal distribution $(\mu_0; \sigma_0^2)$ time i. Then the observations x_i are generated from a normal distribution with parameters $(\mu_i; \sigma_1^2)$. We simulated seven examples and 100,000 mean values been generated in each example and 100,000 observations are generated using the mean values. Finally, the observations are examined by Minitab to calculate the control limits. Results are reported in **Table 9**.

Table 9. Results of the data with variables mean in control chart for Individual Observations

	$\mu_i \sim N(\mu_0, \sigma_0^2)$	$X \sim N(\mu_i, \sigma_1^2)$	$\bar{X}$	α	$\frac{\lvert\alpha_i - 0.0027\rvert}{0.0027}$
1	$\mu_1 \sim N(0; 0.5)$	$X \sim N(\mu_1; 2)$	0	0.00267	0.011
2	$\mu_2 \sim N(0; 1.5)$	$X \sim N(\mu_2; 2.4)$	0.01	0.00263	0.025
3	$\mu_3 \sim N(0; 0.1)$	$X \sim N(\mu_3; 1)$	0	0.00272	0.007
4	$\mu_4 \sim N(0; 0.3)$	$X \sim N(\mu_4; 3)$	0.01	0.00265	0.01
5	$\mu_5 \sim N(0; 1)$	$X \sim N(\mu_5; 1)$	0	0.00271	0.003
6	$\mu_6 \sim N(0; 0)$	$X \sim N(\mu_6; 1)$	0	0.00258	0.04
7	$\mu_7 \sim N(0; 0.8)$	$X \sim N(\mu_7; 2.8)$	0	0.00269	0.003

The results obtained in **Table 9** indicate that probability of first type error is sufficiently close to 0.0027 and results are relatively acceptable. As shown in **Table 9**, the standard error is less than 0.05 in the last column and it represents that the control limit is properly designed. Since the marginal distribution of observations is normally distributed with mean μ_0 and variance $\sigma_0^2 + \sigma_1^2$ and the estimator $\frac{\overline{MR}}{d_2}$ estimates the correct value of standard deviation so the control limits are designed correctly for this control chart.

At the end, it is necessary to mention that the methods used in this article have been previously proposed in various approaches. However, their integration to solve the problem of a process with a variable mean has not been previously presented. In fact, the novel idea of the article is to use two standard deviation estimators that are commonly used in different processes to estimate the parameters in a process with a variable mean. Additionally, the Bayesian theorem is utilized to model the behavior of the process with a variable mean. Since processes with variable means have not been previously studied, it is not possible to compare the performance of the proposed method with them. However, it should be noted that the proposed method is not a new control chart but rather a method for solving the problem of disturbance in the performance of standard control charts in processes with variable means.

10. Conclusion

In some industrial environments, machine setting may fluctuate around a target value and these fluctuations in the production effect on control limits. Therefore, in this research, the effect of variable mean on the control charts is analyzed. First, we show that the Shewhart control chart generates false alarms thus. It is concluded that $\bar{X}$ control chart does not perform properly in such processes. Then this problem is analyzed using the formulation of conditional distribution and the control limits are revised after obtaining the standard deviation of sample mean. Also, it is denoted that this problem does not effect on the R chart thus we cannot use this chart to detect the process with variable mean. A similar result is obtained for S control chart thus the effect of standard deviation of mean values would not effect on both these charts and design of control limits is done correctly for these two control charts. Also, it is denoted that such processes do not effect on the performance of control charts for individual observations. Analyzing the process with variable mean in multivariate control charts is suggested for future researches.

Author contributions

Conceptualization, MSF; methodology, MSF; software, FS; validation, MSF and FS; formal analysis, MSF and FS; investigation, FS; resources, FS; data curation, MSF and FS; writing—original draft preparation, MSFN and FS; writing—review and editing, MSFN and AG;

visualization, FS; supervision, MSF; project administration, MSF. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflict of interest.

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