Bernoulli Collocation Polynomials Algorithm for Calculus of Variational Problems

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Abstract: This paper presents an approximate method that depends on the Bernoulli Polynomials as basic functions. The method is concerned with collocation technique for solving problems in calculus of variation. Some interesting properties of Bernoulli polynomials are used to reduce the original problem to mathematical problem. Some illustrative examples are described to show the applicability of the proposed method.

Keywords: Bernoulli polynomials, variational problems, colloction method

1. Introduction

Several problems appearing in engineering and sciences target to obtain a function of specified functional named the optimal value. Some examples are representing optimal control, optimal shape design, and inverse analysis. Some of them are regarded as variational problems. Many researchers resolved the calculus of variation problem using either direct or indirect methods[1-3].

Polynomials and wavelets are important basis functions that can be used to approximate functions. They are able to deal with many applications in mathematics, physics, and engineering[4-10]. This article studied the Bernoulli polynomials and their important properties[11-13]. Then they are used as basic functions in collocation method to propose an efficient technique for solving variational problems.

2. Bernoulli Polynomials

Generally, Bernoulli polynomials as a function is

$$\frac{te^{z\tau}}{e^\tau - 1} = \sum_0^\infty B_n(\tau)\frac{\tau^n}{n!} \tag{1}$$

The explicit formula for Bernoulli polynomials can be written as

$$B_m(\tau) = \sum_{n=0}^m \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k}(\alpha+k)^m$$

or $B_n(\tau) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k$

The few Bernoulli polynomials can be expressed as

$B_0(\tau) = 1$

$B_1(\tau) = \tau - \dfrac{1}{2}$

$B_2(\tau) = \tau^2 - \tau + \dfrac{1}{6}$

$B_3(\tau) = \tau^3 - \dfrac{3}{2}\tau^2 + \dfrac{1}{2}\tau$

$B_4(\tau) = \tau^4 - 2\tau^2 + \tau^2 - \dfrac{1}{30}$

$B_5(\tau) = \tau^5 - \dfrac{5}{2}\tau^4 + \dfrac{5}{3}\tau^3 - \dfrac{1}{6}\tau$

$$B_6(\tau) = \tau^6 - 3\tau^5 + \frac{5}{2}\tau^4 - \frac{1}{2}\tau^2 + \frac{1}{42}$$

or

$$B_0(\tau) = 1$$

$$B_1(\tau) = \frac{1}{2}(2\tau - 1)$$

$$B_2(\tau) = \frac{1}{6}(6\tau^2 - 6\tau + 1)$$

$$B_3(\tau) = \frac{1}{2}(2\tau^3 - 3\tau^2 + \tau)$$

$$B_4(\tau) = \frac{1}{30}(30\tau^4 - 60\tau^3 + 30\tau^2 - 1)$$

$$B_5(\tau) = \frac{1}{6}(6\tau^5 - 15\tau^4 + 10\tau^3 - \tau)$$

$$B_6(\tau) = \frac{1}{42}(42\tau^6 - 126\tau^5 + 105\tau^4 - 21\tau^2 + 1)$$

Some other properties of Bernoulli polynomials are:

- The Bernoulli numbers are

$$B_1 = -\frac{1}{2} \text{ and } B_{2K+1} = 0, k = 1, 2, \dots$$

$$B_0 = 1, \ B_2 = \frac{1}{2}, \ B_4 = -\frac{1}{30}, \ B_6 = \frac{1}{42}, \ B_8 = -\frac{1}{30}$$

- The Bernoulli Polynomials at zero and one are

$$\dot{B}_2(0) = B_n$$

$$B_n(1) = (-1)^n B_n(0)$$

- The first derivative is $\quad \dot{B}_n(\tau) = nB_{n-1}(\tau)$

3. Bernoulli Collocation Method

In the collocation Bernoulli method, the unknown function is appeared as a linear combination of defined basic functions. The basic functions in many cases are polynomials, orthogonal functions, wavelets functions, or spline approximation. Unknown coefficients are objects that must be computed in order to get an approximate solution.

Suppose the problem of returns the extreme of the functional

$$J[(y(\tau))] = \int_a^b F\big(\tau, y(\tau), \dot{y}(\tau), \dots, y^{(n)}(\tau)\big) dt \qquad (2)$$

Needful condition of $y(t)$ to minimize (or maximize) the functional J in which it must accept the Euler-Lagrange expression

$$F_y - \frac{d}{d\tau}F_{\dot{y}} + \frac{d^2}{d\tau^2}F_{\ddot{y}} + \dots + (-1)^n \frac{d^n}{d\tau^n}F_{y^{(n)}} = 0$$

(3)

That is

$$\frac{\partial F}{\partial y} - \frac{d}{d\tau}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0$$

$$\frac{\partial F}{\partial y} - \frac{d}{d\tau}\left(\frac{\partial F}{\partial \ddot{y}}\right) = 0$$

$$\vdots$$

$$\frac{\partial F}{\partial y^{(n-1)}} - \frac{d}{d\tau}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

Together with suitable boundary conditions, the method will be reduced the original problem to a set of algebraic equations by using the following approach.

Suppose, the solution $y(\tau)$ is approximated as below:

$$y(\tau) = \sum_{i=1}^{N} C_i B_i(\tau)$$

or in matrix form

$$y(\tau) = C^\tau B^\tau \qquad (4)$$

where

$$C^\tau = [C_0 \quad C_1 \quad C_2 \quad \cdots \quad C_N]^\tau$$

$$B^\tau = [B_0(\tau) \quad B_1(\tau) \quad \dots \quad B_N(\tau)]^\tau$$

Differentiate Eq. 4, one can obtain

$$\dot{y}(\tau) = C^\tau \dot{B}(\tau) \qquad (5)$$

where

$$\dot{B}(\tau) = [\dot{B}_0(\tau) \quad \dot{B}_1(\tau) \quad \cdots \quad \dot{B}_N(\tau)]^\tau$$

4. Numerical Examples
4.1 First test example

The first test example is to find the external to the cost functional

$$J = \int_0^1 [\dot{y}^2(\tau) + \tau \dot{y}(\tau)] \, d\tau$$

(6)

with boundary conditions $\quad y(0) = 0 \quad$ and $y(1) = -0.25 \qquad (7)$

here the corresponding Euler equation is

$$\dot{y} = \frac{-1}{2}\tau$$

(8)

Let $\quad\quad y(t) = \sum_{i=0}^{N} C_i B_i(\tau)$

and therefore $\dot{y}(t) = \sum_{i=0}^{N} C_i \dot{B}_i(\tau)$

hence $y(0) = \sum_{i=0}^{N} C_i B_i(0) = 0$

and $\quad y(1) = \sum_{i=0}^{N} C_i B_i(1) = \frac{-1}{4}$

Now one can choose of τ_i points in the interval $[0, 1]$ as $\tau_i = \frac{i}{N}$

Let $N = 3$ that is; $\tau_0 = 0, \ \tau_1 = \frac{1}{3}, \quad \tau_2 = \frac{2}{3} \quad$ and

$\tau_3 = 1$ at $\tau = \tau_1 = \frac{1}{3}$ and at $\tau = \tau_2 = \frac{2}{3}$

The four algebraic equations in four unknown

coefficients are:

$$C_0 B_0(0) + C_1 B_1(0) + C_2 B_2(0) + C_3 B_3(0) = 0$$

$$C_0 \dot{B}_0\left(\frac{1}{3}\right) + C_1 \dot{B}_1\left(\frac{1}{3}\right) + C_2 \dot{B}_2\left(\frac{1}{3}\right) + C_3 \dot{B}_3\left(\frac{1}{3}\right) = -\frac{1}{6}$$

$$C_0 \dot{B}_0\left(\frac{2}{3}\right) + C_1 \dot{B}_1\left(\frac{2}{3}\right) + C_2 \dot{B}_2\left(\frac{2}{3}\right) + C_3 \dot{B}_3\left(\frac{2}{3}\right) = +\frac{1}{3}$$

$$C_0 B_0(1) + C_1 B_1(1) + C_2 B_2(1) + C_3 B_3(1) = \frac{-1}{4}$$

or $C_0 - \frac{1}{2}C_1 + \frac{1}{6}C_2 + 0 = 0$

$$C_0 + C_1 - \frac{1}{3}C_2 - \frac{1}{6}C_3 = -\frac{1}{6}$$

$$C_0 + C_1 + \frac{1}{3}C_2 - \frac{1}{6}C_3 = +\frac{1}{3}$$

$$C_0 + \frac{1}{2}C_1 + \frac{1}{6}C_2 + 0 = \frac{-1}{4}$$

The solution for the above solution will give $y(\tau) = \frac{-1}{4}\tau^2$.

4.2 Second Test Example

The second test problem is to find the extremal of the performance index

$$J(x) = \int_0^1 [\dot{x}^2(\tau) + 1] d\tau$$

(9)

While the both initial and final conditions

$$x(0) = 1 \tag{10}$$

$$x(1) = 2 \tag{11}$$

The corresponding Euler- Lagrange equation is:

$$x + \dot{x} = t + 2$$

(12)

The procedure of the solution is same as example (4.1); the following linear system of equations will be obtained

$$\sum_{i=0}^{3} C_i B_i(0) = 1$$

$$\sum_{i=0}^{3} C_i B_i(1) = 2$$

$$\sum_{i=0}^{3} C_i B_i\left(\frac{1}{3}\right) + \sum_{i=0}^{3} C_i \dot{B}_i\left(\frac{1}{3}\right) = \frac{4}{3}$$

$$\sum_{i=0}^{3} C_i B_i\left(\frac{2}{3}\right) + \sum_{i=0}^{3} C_i \dot{B}_i\left(\frac{2}{3}\right) = \frac{1}{2}$$

The solution $y(\tau) = \tau + 1$ is obtained after solving the above system.

5. Discussion

This works presents an approximate solution for solving calculus of varaitional problems based on collocation method with the Bernoulli polynomials as basic functions. The important properties of Bernoulli polynomials are applied to convert the original problem to the solution of algebraic equations. Two test examples are listed to illustrate the applicability of the method.

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