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Several Inequalities of Gronwall and Their Proofs

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Abstract: It is well known that integral inequalities play a very important role in studying the properties of solutions to ordinary differential equations and integral equations. In 1919, Gronwall established a class of basic integral inequalities when he studied the dependence of differential equations on parameters, which is called Gronwall's inequalities. Gronwall's inequality play a very important role in ordinary differential equations, and it is also an important tool to study the properties of differential equations and integral equation solutions. There are several proofs of Gronwall's inequality, in particular, Agarwal, Deng and Zhang studied the Gronwall-Bellman inequality with multiple nonlinear terms, which made the adaptability of the Gronwall-Bellman inequality widely. Gronwall's inequality has various generalization forms and different proving methods, which is also a good tool for solving many mathematical problems. Different kinds of Gronwall's inequalities and their proofs are discussed in this paper. By researching the induction of Gronwall's inequality forms and their proofs, this paper aims to solve the problems of inequality as much as possible.

Keywords: Gronwall's Inequality; Cauchy Initial Value Problem; RCLL; Continuous Function

1. The Inequality of Gronwall

1.1 The form of inequality of Gronwall

Theorem 1.1 Set $f(t)$ and $g(t)$ be continuous non-real-valued functions on interval $[\alpha,\beta]$, K is non-negative constant, as for $t\in[\alpha,\beta]$, we obtain[1]:

$$f(t)\le K+\int_{\alpha}^{t}f(s)g(s)ds$$

When $t\in[\alpha,\beta]$, we obtain:

$$f(t)\le K\exp(\int_{\alpha}^{t}g(s)ds)$$

2. The inequality of Gronwall with parameters

2.1 The form of inequality of Gronwall with parameters

Theorem 2.1 Set $a(t)$ and $u(t)$ be real-valued and continuous functions greater than or equal to zero on interval $[t_0,t_1]$, assumption to $t\in[\alpha,\beta]$, we obtain[2]:

$$u(t) \le C + \int_{t_0}^{t} a(s)u^{\alpha}(s)ds \qquad (1)$$

C is non-negative constant $\alpha \ne 1$, when $t \in [t_0, t_1]$, we get:

$$u(t) \le [C^{1-\alpha} + (1-\alpha)\int_{t_0}^{t} a(s)ds]^{\frac{1}{1-\alpha}} \qquad (2)$$

Theorem 2.2 Set $a(t)$ and $u(t)$ be real-valued and continuous functions greater than or equal to zero on interval $[t_0, t_1]$, $u(t)$ is monotonic non-decreasing, as for $t \in [t_0, t_1]$,we obtain[3,4]:

$$u(t) \le C + \int_{t_0}^{t} a(s)u^{\alpha}(s)ds, \qquad (3)$$

C is non-negative constant $0 < \alpha < 1$, when $t \in [t_0, t_1]$, we finally obtain:

$$u(t) \le C + u^{\alpha}(t)\int_{t_0}^{t} a(s)ds$$

2.2 The proof of the form of inequality of Gronwall with parameters α

The proof of theorem 2.1:

Consider function $y(t) = \int_{t_0}^{t} a(s)u^{\alpha}(s)ds$ for variable substitution, derivative with respect to t, we obtain:

$$y(t) = a(t)u^{\alpha}(t), y(t_0) = 0 \qquad (4)$$

Both ends of the equation (1) are α th power at the same time, and then multiply by $\alpha(t) \ge 0$, put it in (3), we can get:

$$y(t) \le a(t) \quad [C + \int_{t_0}^{t} a(s)u^{\alpha}(s)ds]^{\alpha} = a(t) \quad [C + y(t)]^{\alpha}\text{ ,}$$

Consider the integration factor of first order ordinary differential equation

$y(t) = a(t) \quad [C + y(t)]^{\alpha}$

$$\frac{1}{[C + y(t)]^{\alpha}} \ge 0,$$

Both ends of the above formula multiply by the integration factor at the same time, we can get:

$$\frac{y(t)}{[C + y(t)]^{\alpha}} \le a(t)$$

And integrate both sides of this equation from t_0 to t, we obtain:

$$\frac{1}{1-\alpha}[C+y(t)]^{1-\alpha}-\frac{C^{1-\alpha}}{1-\alpha}\le\int_{t_0}^{t}a(s)ds$$

When $\alpha<1$, $u(t)\le C+y(t)$, we know that $g(x)=\frac{1}{1-\alpha}x^{1-\alpha}$ is increment function at $x>0$, and

$$\frac{u^{1-\alpha}(t)}{1-\alpha}\le\frac{1}{1-\alpha}[C+y(t)]^{1-\alpha}$$

so $\frac{u^{1-\alpha}(t)}{1-\alpha}\le\frac{C^{1-\alpha}}{1-\alpha}+\int_{t_0}^{t}a(s)ds$.

then $u(t)\le[C^{1-\alpha}+(1-\alpha)\int_{t_0}^{t}a(s)ds]^{\frac{1}{1-\alpha}}$.

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so $\frac{u^{1-\alpha}(t)}{1-\alpha}\le\frac{C^{1-\alpha}}{1-\alpha}+\int_{t_0}^{t}a(s)ds$.

then $u^{1-\alpha}(t)\ge C^{1-\alpha}+(1-\alpha)\int_{t_0}^{t}a(s)ds$.

3. The maximum term exists in the nonlinear Gronwall integral inequality

3.1 The form of the maximum term exists in the nonlinear Gronwall integral inequality

The maximum term exists in the linear Gronwall integral inequality was discussed by Hristova and Stefanova in 2010.

$$\begin{cases} u(t)\le k(t)+f(t)\int_{t_0}^{t}[p(s)u(s)+q(s)\max_{\xi\in[s-h,s]}u(\xi)]ds \\ \qquad +g(t)\int_{\alpha(t_0)}^{\alpha(t)}[a(s)u(s)+b(s)\max_{\xi\in[s-h,s]}u(\xi)]ds,\quad t\in[t_0,T). \qquad (5) \\ u(t)\le\Phi(t),t\in[\alpha(t_0)-h,t_0]. \qquad (6) \end{cases}$$

And the maximum term exists in the nonlinear Gronwall integral inequality was discussed by Hristova in 2012, we consider $k(t)=k$ first, then

$$\begin{cases} u(t) \le k + \int_{t_0}^{t} [p_1(t,s)g(u(s)) + p_2(t,s)g(\max_{\xi\in[s-h,s]} u(\xi)]ds \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)} [p_3(t,s)g(u(s)) + p_4(t,s)g(\max_{\xi\in[s-h,s]} u(\xi)]ds, \quad t \in [t_0, T). & (7) \\ u(t) \le \Phi(t), \quad t \in [t_0 - h, t_0]. & (8) \end{cases}$$

We assume that:

(H_1) Nonnegative continuously differentiable and monotonically nondecreasing function $a(t)$ in $[t_0, T]$, and $a(t) \le t$;

(H_2) Non-negative continuous function $p_1(t,s)$ and $p_2(t,s)$ in $[t_0,T)\times[t_0,T)$, non-negative continuous increasing function $p_3(t,s)$ and $p_4(t,s)$ in $[t_0,T)\times[\alpha(t_0),T)$;

(H_3) Non-negative continuous function $\Phi(t)$ is defined in $t \in [t_0 - h, t_0]$, and $\Phi(t) \ge k$;

(H_4) Non-negative continuous increasing function $g(t)$ is defined in R_+; and $0 \le t_0 \le T \le \infty$, h is non-negative.

Theorem 3.1 If (H_1) - (H_4) is established at the same time, $u \in C([\alpha(t_0) - h, T), R_+)$ also established, and satisfy the inequality (7) and (8), then for any $t \in (t_0, t_1)$, we get an estimator of the unknown function in inequality (7)[5,6]:

$$u(t) \le G^{-1}(G(k) + \int_{t_0}^{t} [\tilde{p}_1(t,s) + \tilde{p}_2(t,s)]ds + \int_{\alpha(t_0)}^{\alpha(t)} [\tilde{p}_3(t,s) + \tilde{p}_4(t,s)]ds). \quad (9)$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \quad r_0 > 0, \quad (10)$$

G^{-1} is the inverse of G: $\tilde{p}_i(t,s) = \max_{\tau\in[t_0,t]} p_i(\tau,s) \quad i = 1,2,3,4;$ and:

$$t_1 = \sup\{\tau \ge t_0 : G(k) + \int_{t_0}^{t} [\tilde{p}_1(t,s) + \tilde{p}_2(t,s)]ds + \int_{\alpha(t_0)}^{\alpha(t)} [\tilde{p}_3(t,s) + \tilde{p}_4(t,s)]ds \in Dom(G^{-1}), t \in [t_0, \tau) \}.$$

3.2 The proof of the maximum term exists in the nonlinear Gronwall integral inequality

The proof of theorem 3.1: For any non-negative $t \in [t_0, T), p_i(t,s) \le \tilde{p}_i(t,s), \tilde{p}_i(t,s)$ and monotonic non-decreasing function with respect to t, we obtain (7) as:

$$u(t) \le k + \int_{t_0}^{t} [\tilde{p}_1(t,s)g(u(s)) + \tilde{p}_2(t,s)g(\max_{\xi\in[s-h,s]} u(\xi)]ds +$$
$$\int_{\alpha(t_0)}^{\alpha(t)} [\tilde{p}_3(t,s)g(u(s)) + \tilde{p}_4(t,s)g(\max_{\xi\in[s-h,s]} u(\xi)]ds$$

Any $t_2 \in [t_0, T)$, to $t \in [t_0, t_2)$, the following inequality holds:

$$u(t) \le k + \int_{t_0}^{t} [\tilde{p}_1(t_2,s)g(u(s)) + \tilde{p}_2(t_2,s)g(\max_{\xi\in[s-h,s]} u(\xi)]ds +$$
$$\int_{\alpha(t_0)}^{\alpha(t)} [\tilde{p}_3(t_2,s)g(u(s)) + \tilde{p}_4(t_2,s)g(\max_{\xi\in[s-h,s]} u(\xi)]ds .$$

We denote $z : [\alpha(t_0) - h, t_2) \to R_+$, and

$$z(t) = \begin{cases} k, t \in [\alpha(t_0) - h, t_0], \\ k + \int_{t_0}^{t} [\tilde{p}_1(t_2,s)g(u(s)) + \tilde{p}_2(t_2,s)g(\max_{\xi\in[s-h,s]} u(\xi)]ds \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)} [\tilde{p}_3(t_2,s)g(u(s)) + \tilde{p}_4(t_2,s)g(\max_{\xi\in[s-h,s]} u(\xi)]ds, \ t \in [t_0, t_1). \end{cases}$$

$z(t)$ is monotonically non-decreasing and satisfies u(t) ≤ z(t) on $[\alpha(t_0) - h, t_2)$.

We know that:

$$\max_{\xi\in[t-h,t]} z(s) = z(t), t \in [\alpha(t_0), t_2).$$

As for $t \in [t_0, t_2)$, we obtain:

$$z(t) \le k + \int_{t_0}^{t} [\tilde{p}_1(t_2,s)g(z(s)) + \tilde{p}_2(t_2,s)g(\max_{\xi\in[s-h,s]} z(\xi)]ds +$$

$$\int_{\alpha(t_0)}^{\alpha(t)} [\tilde{p}_3(t_2,s)g(z(s)) + \tilde{p}_4(t_2,s)g(\max_{\xi\in[s-h,s]} z(\xi)]ds$$

$$\le k + \int_{t_0}^{t} [\tilde{p}_1(t_2,s) + \tilde{p}_2(t_2,s)]g(z(s))ds +$$

$$\int_{\alpha(t_0)}^{\alpha(t)} [\tilde{p}_3(t_2,s) + \tilde{p}_4(t_2,s)]g(z(s))ds. \tag{11}$$

Take the derivative of (11):

$$(z(t))' \le [\tilde{p}_1(t_2,t) + \tilde{p}_2(t_2,t)]g(z(t)) + [\tilde{p}_3(t_2,\alpha(t)) + \tilde{p}_4(t_2,\alpha(t))](\alpha(t))' \le$$

$$g(z(t))[\tilde{p}_1(t_2,t)+\tilde{p}_2(t_2,t)+(\tilde{p}_3(t_2,\alpha(t))+\tilde{p}_4(t_2,\alpha(t))(\alpha(t))' \quad (12)$$

From equations (10) and (12), we can get:

$$\frac{d}{dt}G(z(t))=\frac{(z(t))}{g(z(t))}\le \tilde{p}_1(t_2,t)+\tilde{p}_2(t_2,t)+(\tilde{p}_3(t_2,\alpha(t))+\tilde{p}_4(t_2,\alpha(t))(\alpha(t)). \quad (13)$$

Integrate from $t_0 \to t$ against (13) ends, and apply the exchange decree $\eta=\alpha(s)$, we can get:

$$G(z(t))\le G(k)+\int_{t_0}^{t}[\tilde{p}_1(t_2,\eta)+\tilde{p}_2(t_2,\eta)]d\eta+\int_{\alpha(t_0)}^{\alpha(t)}[\tilde{p}_3(t_2,\eta)+\tilde{p}_4(t_2,\eta)]d\eta.$$

therefore,

$$G(z(t))\le G(k)+\int_{t_0}^{t}[\tilde{p}_1(t_2,s)+\tilde{p}_2(t_2,s)]ds+\int_{\alpha(t_0)}^{\alpha(t)}[\tilde{p}_3(t_2,s)+\tilde{p}_4(t_2,s)]ds.$$

Since G^{-1} is an increasing function, we finally obtain:

$$z(t)\le G^{-1}(G(k)+\int_{t_0}^{t}[\tilde{p}_1(t_2,s)+\tilde{p}_2(t_2,s)]ds+\int_{\alpha(t_0)}^{\alpha(t)}[\tilde{p}_3(t_2,s)+\tilde{p}_4(t_2,s)]ds)$$

and $u(t)\le G^{-1}(G(k)+\int_{t_0}^{t}[\tilde{p}_1(t_2,s)+\tilde{p}_2(t_2,s)]ds+\int_{\alpha(t_0)}^{\alpha(t)}[\tilde{p}_3(t_2,s)+\tilde{p}_4(t_2,s)]ds)$ can be obtained from $u(t)\le z(t)$,

From the arbitrariness of t_2, set $t_2=t$, get (9), the proof is completed.

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