Non-smooth Waves and Anti-solitons in the General Degasperis-Procesi Model

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Abstract: We consider the general Degasperis-Procesi model of shallow water out-flows, which generalizes the list of famous equations: KdV, Benjamin-Bona-Mahony, Camassa-Holm, and Degasperis-Procesi. Our objective is the construction of self-similar solutions of this equation. Along with the standard waves (peakons and cuspons) we present a new type of solutions (that we call "twins") which are a combination of solitons and cuspons. We demonstrate also the wave-kind dependence on the amplitude for the waves (solitons, peakons, cuspons, and twins) with positive and negative amplitudes.

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1. Introduction

We consider a unidirectional approximation of the shallow water system called the "general Degasperis-Procesi" (gDP) model ([1], 1999):

$$\frac{\partial}{\partial t}\left\{u-\alpha^2\varepsilon^2\frac{\partial^2 u}{\partial x^2}\right\} + \frac{\partial}{\partial x}\left\{c_0u+c_1u^2-c_2\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2+\varepsilon^2\left(\gamma-c_3u\right)\frac{\partial^2 u}{\partial x^2}\right\}=0,\ x\in\mathbb{R}^1,\ t>0. \qquad (1)$$

Here α, c_0 , …, γ are real parameters and ε characterizes the dispersion. The constants $\alpha \geq 0$, and $\gamma \geq 0$ are associated with different characters of the dispersion manifestation. In the Green-Naghdi approximation the restriction $\alpha + \gamma = 1/6$ is required. The equation (1) terms with $c_2 \geq 0$ and $c_3 \geq 0$ can be treated as representations of nonlinear dispersion. In the Camassa-Holm approximation $c_2 + c_3 > 0$.

This six parametric family of third order conservation laws generalizes a list of famous equations. Indeed:

1. If we set $\alpha = c_2 = c_3 = 0$ then we obtain the famous KdV equation, whereas for $\gamma = c_2 = c_3 = 0$ Eq.(1) is the well-known Benjamin-Bona- Mahony (BBM) equation ([2], 1972)

2. Preserving in (1) the nonlinear dispersion terms and setting $c_2 = c_3/2$, $c_1 = 3c_3/2\alpha^2$, and $\gamma = 0$ we obtain the Camassa-Holm (CH) equation ([3], 1993):

$$\frac{\partial}{\partial t}\left\{u-\alpha^2\varepsilon^2\frac{\partial^2 u}{\partial x^2}\right\} + \frac{\partial}{\partial x}\left\{c_0u+c_1u^2-\varepsilon^2\frac{c_3}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2+2u\frac{\partial^2 u}{\partial x^2}\right\}\right\}=0. \qquad (2)$$

3. In the case $c_2 = c_3$, $c_1 = 3c_3/2\alpha^2$, and $c_0 = \gamma = 0$ (1) is the Degasperis- Procesi (DP) equation ([1], see also[4] and references therein):

The KdV and BBM equations are essentially different. Both of them have soliton-type traveling wave solutions,

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however, KdV solitons collide elastically: they pass through each other preserving the shapes and velocities, whereas BBM "solitons" change after the interaction and an oscillatory tail is generated[5].

Next, for the first view the CH (6) and DP (7) equations are quite similar: the difference consists of the relation between the coefficients c_0, c_2, and c_3 only. However, it should be emphasized that these equations have truly different properties:

- if $c_0 > 0$, the Camassa-Holm equation has smooth soliton solution

$$u(x,t) = A\omega\big((x-Vt)/\varepsilon, A\big), \quad \omega(\tau,\cdot) \in C^\infty(\mathbb{R}^1), \quad \lim_{\tau\to\pm\infty}\omega(\tau,\cdot)=0, \quad (4)$$

- if $c_0 = 0$, the Camassa-Holm equation has continuous but non-smooth traveling wave solutions $u(\eta)|_{\eta=(x-Vt)/\varepsilon}$ called "peakons" if their first derivative are bounded or "cuspons" if the first derivatives are unbounded (at the point $\eta_0 = 0$).

- the Degasperis-Procesi equation, under the condition $u \to 0$ as $x \to \infty$, doesn't admit smooth traveling wave solutions.

The cited particular cases of the equation (1) have been studied intensively (see e.g.[5-16] and references therein). As for the general family (1), the first systematic results have been obtained recently in the papers [17, 18], dedicated, on the whole, to smooth soliton type solutions. The main objective of the present paper is the construction of non-smooth traveling wave solution of (1). Next, completing[18], we describe the set of possible traveling wave solutions of (1) which vanish at infinity.

Similarly to CH and DP equations, it is natural to suppose the existence of peakons and cuspons for the general DP equation (1) also. Indeed, we prove that this is true, but under some conditions. Next, it turns out that some non-integrable versions of (1) admit a new kind of solutions which are of the soliton type but they have a non-bounded first derivative at points $\pm\eta_0 \neq 0$.

We call this soliton-cuspon combination "twins" (see below **Figure 1** and **Figure 2**).

To justify continuous functions as well-defined solutions of gDP equation we transform (1) into the following divergent form:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left\{c_0 u + c_1 u^2 - (c_2 - c_3)\left(\varepsilon\frac{\partial u}{\partial x}\right)^2\right\} \quad (5)$$
$$= \varepsilon^2\frac{\partial^2}{\partial x^2}\left\{\alpha^2\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\gamma u - \frac{c_3}{2}u^2\right)\right\},$$

and note that all terms here are well defined not for smooth functions only,

but for distributions such that $(u(\eta)'_\eta)^2 \in \mathfrak{D}'(\mathbb{R}^1)$ also. Note that the Degasperis-Procesi equation with $c_2 = c_3$ has more singular solutions, however we will not consider here this well studied case, see e.g.[1,6,11-16].

Concerning the solution construction, we use an approach based on the algebraic point of view. Indeed, non-classical traveling waves $u = u(x - Vt)$ of (1) should be distributions such that $(u(\eta)'_\eta)^2 \in \mathfrak{D}'(\mathbb{R}^1)$, in other words $u(\eta)$ and $u(\eta)'_\eta$ should belong to a subalgebra in $\mathfrak{D}'(\mathbb{R}^1)$. In fact, there exist only three subalgebras in $\mathfrak{D}'(\mathbb{R}^1)$. The first one has the generators $\{1,\ H(\eta)\}$, where 1 denotes the space of smooth functions and $H(\eta)$ is the Heaviside function: $H(\eta) = 0$ for $\eta < 0$, and $H(\eta) = 1$ for $\eta > 0$. The Heaviside function is associated with the sequence

$$\ldots, \quad \eta_+, \quad H(\eta), \quad \delta(\eta), \quad \delta'(\eta), \ldots, \quad (6)$$

where $\eta_+ = \eta H(\eta)$; $\delta(\eta)$ and $\delta'(\eta), \ldots$ are the Dirac delta-function and its derivatives. This subalgebra allows us to construct peakon-type solutions.

The second subalgebra has the generators $\{1, \eta_+^\lambda\}$, where $\lambda \in (0,1)$ (see e.g. [19]). Respectively, the distribution η_+^λ is associated with the sequence

$$\ldots, \quad \eta_+^{\lambda+2}, \quad \eta_+^{\lambda+1}, \quad \eta_+^{\lambda}, \quad \eta_+^{\lambda-1}, \quad \eta_+^{\lambda-2}, \ldots, \quad (7)$$

and allows us to construct both cuspon-type and twins solutions.

To obtain the third subalgebra we should treat ε as a small parameter and associate distributions with asymptotic series with respect to ε with coefficients from $\mathfrak{D}'(\mathbb{R}^1)$. In the framework of such interpretation, solitons, in the leading term with respect to ε, are associated with the so-called "function": $\big(\varepsilon\delta(\eta), \psi(\eta)\big) = \varepsilon\psi(0)$. Since powers of solitons are associated with $\varepsilon\delta$-function again, $\{1,\ \varepsilon\delta(\eta)\}$ forms the subalgebra. This treatment of solitons had allowed us to describe the motion of perturbed solitons[20,21] and, not so long ago, collision of solitons for essentially non-integrable equations[22-25].

In what follows we assume that

$$\gamma \geq 0, \quad \alpha \geq 0, \quad \gamma+\alpha>0, \quad c_0 \geq 0, \quad c_k>0, \quad k=1,2,3, \qquad (8)$$

and treat $\varepsilon \neq 0$ as a fixed parameter.

The paper contents is the following: In Section 2 we present the construction of peakons and obtain sufficient conditions of their existence. The cuspon and twins cases are considered in Sections 3 and 4. Section 5 is devoted to waves with negative amplitudes. In Conclusion we describe the set of possible traveling wave solutions of Eq. (1).

2. Peakons

To construct a peakon solitary wave let us define the notation

$$[f]=f_{+}(\eta)-f_{-}(\eta), \quad [f]|_0=f_{+}(\eta)|_{\eta \to +0}-f_{-}(\eta)|_{\eta \to -0}, \qquad (9)$$

for arbitrary functions $f_{\pm}(\eta)$. Next we write the ansatz

$$u(x,t,\varepsilon)=A\{\omega_{-}(\eta)+[\omega] H(x-Vt)\}|_{\eta=\beta(x-Vt)/\varepsilon}. \qquad (10)$$

Here $\omega_{\pm}=\omega_{\pm}(\eta)$ are functions such that:

$$\omega_{\pm}|_{\eta=\pm 0}=1, \quad \omega_{\pm}(\eta) \to 0 \quad as \quad \eta \to \pm\infty, \qquad (11)$$

$$\omega_{\pm}(\eta) \in C^{\infty}(\mathbb{R}_{\pm}^{1}), \qquad (12)$$

the amplitude $A>0$ is a free parameter, and the velocity $V=V(A)$ should be determined. To simplify formulas we define the scale

$$\beta=\sqrt{c_1(c_2+c_3)}\ /c_3$$

Obviously, (11) implies that $[\omega]|_0=0$, however, to obtain a peakon we should suppose

$$[\omega']|_{\eta=0} \neq 0, \qquad (13)$$

where prime denotes the derivative with respect to η. We assume also that the functions $\omega_{\pm}$ are extended on $\mathbb{R}_{\mp}^{1}$ in a smooth manner.

Note now that $H^2=H$, thus

$$u^2(x,t,\varepsilon)=A^2\{\omega_{-}^2(\eta)+[\omega^2] H(x-Vt)\}|_{\eta=\beta(x-Vt)/\varepsilon}. \qquad (14)$$

Furthermore,

$$\begin{aligned} \varepsilon \frac{\partial u(x,t,\varepsilon)}{\partial x} &= A\beta\{\omega_{-}'(\eta)+[\omega'] H(x-Vt)\}|_{\eta=\beta(x-Vt)/\varepsilon}, \qquad (15)\\ \varepsilon^3 \frac{\partial^3 u(x,t,\varepsilon)}{\partial x^3} &= A\beta^3\{\omega_{-}'''(\eta)+[\omega'''] H(x-Vt)\\ &\quad +\frac{\varepsilon}{\beta}[\omega'']|_0 \delta(x-Vt)+\frac{\varepsilon^2}{\beta^2}[\omega']|_0 \delta'(x-Vt)\}|_{\eta=\beta(x-Vt)/\varepsilon}. \end{aligned}$$

Substituting (14), (15), and similar relation for u_x^2 into (5), we obtain a linear combination of $H(\eta)$, $1-H(\eta)$, $\delta(\eta)$, and $\delta'(\eta)$ functions. Now, in order to simplify formulas, we rescale the functions $\omega_{\pm}$

$$W_{\pm}=p\omega_{\pm}, \quad p=c_3 A/(\gamma+\alpha^2 V), \qquad (16)$$

and define the notation

$$r=c_3/(c_2+c_3), \quad q=c_3(V-c_0)/\big(c_1(\gamma+\alpha^2 V)\big). \qquad (17)$$

Then the result of substitution of (10) into (5) can be easily transformed to the following form:

$$\begin{aligned} &\{\mathfrak{W}_{-}+[\mathfrak{W}]H\}+\frac{\varepsilon^2}{\beta^2}\Big\{[W']|_0-\frac{1}{2}[(W^2)']|_0\Big\}\delta' \\ &\quad +\frac{\varepsilon}{\beta}\Big\{[W'']|_0-\frac{c_2-c_3}{c_3}[(W')^2]|_0-\frac{1}{2}[(W^2)'']|_0\Big\}\delta=0, \qquad (18) \end{aligned}$$

where

$$\mathfrak{W}\pm=\frac{d}{d\eta}\Big\{rW_{\pm}^2-rqW_{\pm}+W_{\pm}''-\frac{c_2-c_3}{c_3}(W_{\pm}')^2-\frac{1}{2}\big(W_{\pm}^2\big)''\Big\}. \qquad (19)$$

Recall that the distributions H, δ, and δ' are linearly independent. Thus by virtue of (11) and (18) we deduce that:

$$(1-p)[W']|_0=0, \quad (1-p)[W'']|_0-\frac{c_2}{c_3}[(W')^2]|_0=0. \qquad (20)$$

Clearly, for peakons we conclude that:

$$p=1, \quad W_{-}(\eta)=W_{+}(-\eta) \quad \text{for} \quad \eta \leq 0. \qquad (21)$$

Consequently, (18) implies now the problems

$$(1-W_\pm)\frac{d^2W_\pm}{d\eta^2}=\frac{1-r}{r}\left(\frac{dW_\pm}{d\eta}\right)^2+r(qW_\pm-W_\pm^2),\quad \eta\in\mathbb{R}^1_\pm, \tag{22}$$

$$W_\pm|_{\eta=\pm0}=1,\quad W_\pm(\eta)\to 0\quad as\quad \eta\to\pm\infty. \tag{23}$$

It is important to note that when $\alpha>0$, then the equality $p=1$ defines the velocity

$$V=\alpha^{-2}(c_3A-\gamma). \tag{24}$$

Thus, all coefficients in (22) are uniquely defined in this case. On the contrary, if $\alpha=0$, we should fix the amplitude $A=\gamma/c_3$. Respectively, the coefficient $q=q(V)$ remains indeterminated.

Furthermore, to analyze the equation (22) let us change the variable

$$W_\pm(\eta)=1-g_\pm(\eta)^r. \tag{25}$$

We take into account the property of being even, $g(-\eta)=g(\eta)$. Then we eliminate the first derivatives from the model equations (22) and pass to the boundary problem

$$\frac{d^2g_+}{d\eta^2}=g_+-(2-q)g_+^{1-r}+(1-q)g_+^{1-2r},\quad \eta\in(0,\infty), \tag{26}$$

$$g_+\big|_{\eta=0}=0,\quad g_+|_{\eta\to\infty}=1. \tag{27}$$

Now we integrate (26), use the second condition in (27), and obtain the Cauchy problem for the first order ODE.

$$\frac{dg_+}{d\eta}=\sqrt{F(g_+,q)},\quad \eta\in(0,\infty), \tag{28}$$

$$g_+|_{\eta=0}=0. \tag{29}$$

Here

$$F(g,q)=g^2-2\frac{2-q}{2-r}g^{2-r}+\frac{1-q}{1-r}g^{2-2r}-C(q), \tag{30}$$

$$C(q)=r(r-q)/\{(1-r)(2-r)\}. \tag{31}$$

Let us note that

$$\omega_\pm'=\mp rg_\pm^{r-1}\sqrt{F(g_\pm,q)}. \tag{32}$$

Thus, for $g_\pm(0)=0$ the derivative $\omega'_\pm(0)$ is bounded if and only if $C(q)=0$.

This implies the condition

$$q=r\quad\Leftrightarrow\quad\frac{V-c_0}{\gamma+\alpha^2V}=r\frac{c_1}{c_3}. \tag{33}$$

Therefore

$$F(g,r)=g^2(g^{-r}-1)^2. \tag{34}$$

Integrating we obtain the basic peakon solution:

$$\omega_\pm=\exp(\mp r\eta). \tag{35}$$

Let us focus on the condition of the existence of peakons.

1. Suppose $\alpha>0$. In accordance with (24) we define $V=V(A)$ and conclude from (33) that

if $c_3>\alpha^2rc_1$, then $A=A^*$, where

$$A^*=\frac{\gamma_\alpha}{c_3-\alpha^2rc_1},\quad \gamma_\alpha=\gamma+\alpha^2c_0. \tag{36}$$

Consequently, if $\gamma_\alpha>0$, then the peakon exists with uniquely defined V and $A>0$. At the same time, if $\gamma_\alpha=0$, that is for $\gamma=0$ and $c_0=0$, the peakon can't have positive amplitude. Conversely,

$$\text{if}\quad c_3=\alpha^2rc_1\quad\text{and}\quad\gamma_\alpha=0, \tag{37}$$

then the peakon can be of arbitrary amplitude. Finally, if $c_3\le\alpha^2rc_1$ and $\gamma_\alpha>0$, then peakons with positive amplitudes don't exist.

2. Suppose $\alpha=0$. In accordance with (24) and (33) we conclude that the peakon exists with uniquely defined V and $A>0$:

3. Cuspons

To construct a cuspon-type traveling wave we will use the ansatz (10), (11), (16), (25) again. However, according to (7) and contrary to (12) we assume

$$\omega_\pm(\eta)=1-g_\pm^r(\eta),\quad g_\pm(\eta)\to1\quad\text{as}\quad\eta\to\pm\infty, \tag{39}$$

$$g_\pm(0)=0,\quad g_\pm'(0)=\text{const}\neq0, \tag{40}$$

where $g(\eta)\in C^\infty(R^1)$, and the degree r is defined in (17). In view of (5) we suppose

$$r>1/2\quad\Leftrightarrow\quad c_3>c_2. \tag{41}$$

Repeating the construction from Sec.2 we pass to the equation (28) again. However, for solutions with unbounded derivative at zero we should assume now that $C(q) < 0$. Thus, we obtain the conditions of a cuspon-wave existence:

$$\frac{c_3 A}{\gamma + \alpha^2 V} = 1, \quad q > r \quad \Leftrightarrow \quad \frac{V - c_0}{\gamma + \alpha^2 V} > r\frac{c_1}{c_3}. \tag{42}$$

Again, there are different possibilities:

1. Suppose $\alpha > 0$. Then the velocity V is defined by (24). Next,

$$\text{if} \quad c_3 > \alpha^2 r c_1, \quad \text{then} \quad A > A^*. \tag{43}$$

Consequently, if $\gamma_\alpha > 0$, then there exists the family of cuspons with amplitudes A under the restriction (43). If $\gamma_\alpha = 0$, the cuspon can be of arbitrary positive amplitude. Conversely, cuspons don't exist if

$$c_3 \le \alpha^2 r c_1. \tag{44}$$

2. Suppose $\alpha = 0$. Then, instead of (38) we obtain

$$A = \gamma / c_3, \quad V > c_0 + r c_1 A. \tag{45}$$

This means that we can define initial data for (5) like cuspons with fixed amplitude but with different rates of decrease due different values of $q > r$ in (28). Respectively, the traveling wave will propagate with the velocity

$$V = c_0 + q c_1 A. \tag{46}$$

4. Twins

Let us use the ansatz formally like (10), (11), (16), (25) again, but contrary to (12), (13), and (40) we suppose

$$W_\pm|_{\eta=0} = p > 0, \quad W'_\pm|_{\eta=0} = 0. \tag{47}$$

If $C > 0$, that is $q < r$, then $g(\eta) \in (0,1)$, and the associated solution is the classical soliton [17, 18]. However, for $C < 0$ the range of g can include negative values, therefore the solution can be singular at a point $\eta_0 > 0$ where $g(\eta_0) = 0$. Let us consider this case under the condition $q > r$.

Assume first that for some i, $k \in Z_+$

$$2(k-i)c_3 = (2i+1)c_2 \quad \text{for} \quad 0 < i < k, \quad \text{and} \quad c_3 > c_2. \tag{48}$$

Then $r = \frac{2i+1}{2k+1} > 1/2$ so that g^r is well defined over $\mathbb{R}^1$ and $(-g)^r = -g^r$.

Turn next to the case in which the range of $g(\eta)^r$ includes negative values. We come back to the equation (28) and consider the right-hand side F(g,q). It is easy to calculate the first derivative

$$F'_g = (1-q)\, g\, (g^{-r} - 1)\Big(g^{-r} - (1-q)^{-1}\Big).$$

Thus the left critical point $g_0 = (1-q)^{1/r}$ is negative for $q > 1$. Furthermore,

$$F(g_0, q) = -\frac{r}{2-r}\Big\{\Big((q-1)^{2/r} - 1\Big) + \frac{q-1}{1-r}\Big((q-1)^{2(1-r)/r} - 1\Big)\Big\}. \tag{49}$$

Obviously, $F(g_0, q) < 0$ when $q > 2$, whereas $F(0,q) > 0$. Moreover,

$F''_{qq} > 0$. Thus, the condition

$$q > 2 \tag{50}$$

guaranties the existence of a point $g_* < 0$ such that

$$F(g_*, q) = 0, \quad F(g,q) > 0 \quad \text{for} \quad g \in (g_*, 1). \tag{51}$$

Now let us consider the Cauchy problem consisting of the equation (28) and the initial data

$$g_+|_{\eta=0} = g_*. \tag{52}$$

In view of the property $g_+ \to 1$ as $\eta \to \infty$ and (52), we obtain the existence of the required point $\eta_0 > 0$ as well as the existence of the even extension of $g_+(\eta)$ on $\mathbb{R}^1_-$, which is smooth near the point $\eta = 0$. Now we define

$$p = 1 - g_*^r, \quad W_-(\eta) = W_+(-\eta) \quad \text{for} \quad \eta < 0, \tag{53}$$

and justify the satisfiability of (47) for $q > 2$. It remains to note that equation (32) yields

$$W'_+|_{\eta\to\eta_0} \approx -(\eta - \eta_0)^{r-1} \to -\infty.$$

Remark 1. The equalities $1 - g_*^r = p(A, V)$ and

(17) determine the relations between A, V, q, and g_*. Indeed, let $\alpha > 0$. Then (16), (17), and (53) imply

$$V = \frac{1}{\alpha^2}\left\{\frac{c_3 A}{1-g_*^r} - \gamma\right\}, \quad q = \frac{c_3}{\alpha^2 c_1} - \frac{\gamma_\alpha}{\alpha^2 A c_1}(1-g_*^r). \tag{54}$$

This implies the equation $F(g_*, q(g_*, A)) = 0$, which is resolvable for $q > 2$. Respectively, we obtain the one-parametric family of twins under the condition (50), that is for $A > 0$:

$$(c_3 - 2\alpha^2 c_1)A > \gamma_\alpha(1-g_*^r). \tag{55}$$

If $\alpha = 0$, then we treat $q > 2$ as a free parameter, solve the equation $F(g_*, q) = 0$ and obtain the one-parametric family of twins again. Now the shape of each twins depends on the q-value, and it propagates with the individual velocity and amplitude

$$V = c_0 + q\gamma c_1/c_3, \quad A = \gamma(1-g_*^r)/c_3. \tag{56}$$

Remark 2. Notice the difference between cuspons and twins. Indeed, cuspons are associated with the "even combination" of the distributions $\eta_\pm^r$: locally $u \approx Ap^{-1}(1 - c(\eta_+^r + \eta_-^r))$, where $c = (-C)^{r/2}$ and $C < 0$. At the same time, twins are associated with the "odd combination" of the distributions of the type $\eta_\pm^r$: locally $u|_{\eta>0} \approx Ap^{-1}(1 - c((\eta-\eta_0)_+^r - (\eta-\eta_0)_-^r)$

Example (J. Noyola Rodriguez)

Let

$$\gamma = 2, \quad \alpha = 1/3, \quad c_0 = c_1 = 1, \quad c_2 = 2, \quad c_3 = 3, \quad \varepsilon = 0,1. \tag{57}$$

Then $r = 3/5$ and $c_3 > 2\alpha^2 c_1$. Under the choice

$$q = 2,1 \tag{58}$$

we solve the equation (51) and define the required root $g_* = -0.652493832$. Respectively, we find the wave amplitude $A = 1.35366126$ Now we apply the Runge-Kutta method for the problem (28), (52) and obtain the twins depicted in **Figure 1** and **Figure 2**.

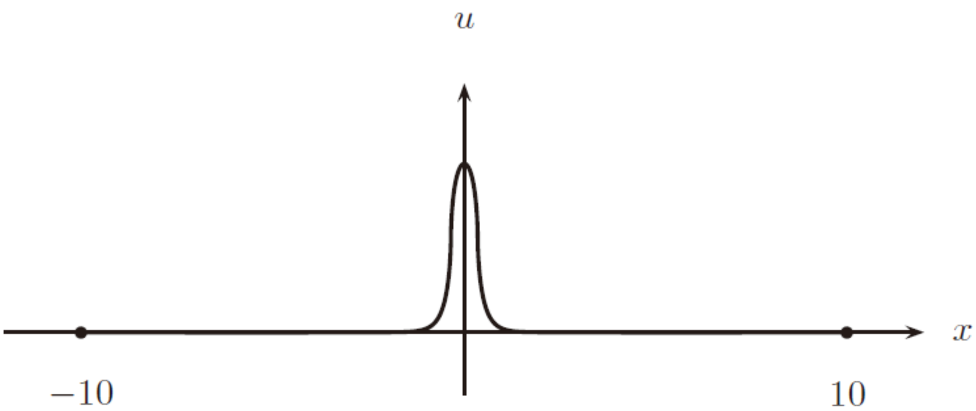

Figure 1: The twins function in the case (57), (58).

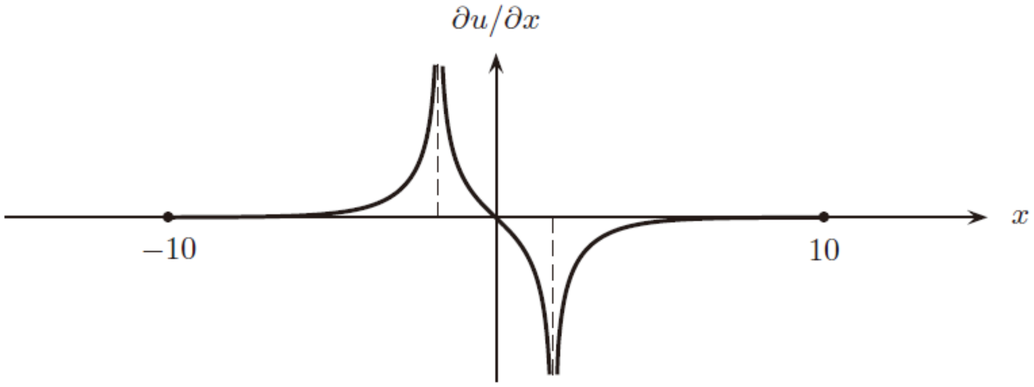

Figure 2: The twins derivative in the case (57), (58).

5. Waves with negative amplitudes

Contrary to the KdV equation, the model (1) admits self-similar solutions with negative amplitudes. Let us establish the conditions for their existence.

Peakons. By virtue of the equalities $p = 1$ and (16), the assumption $A < 0$ requirers $\alpha > 0$. Next we define the velocity in the form of (24) and pass to the equation $q = r$. Obviously, its solution $A = A^* < 0$ exists if and only if

$$c_3 < \alpha^2 r c_1. \tag{59}$$

Cuspons. We assume $\alpha > 0$. Next, the inequality q>r implies now:

$$\text{if} \quad c_3 < \alpha^2 r c_1 \quad \text{then} \quad A \in (A^*, 0), \tag{60}$$

$$\text{if} \quad c_3 \geq \alpha^2 r c_1 \quad \text{then} \quad A < 0, \tag{61}$$

where $A^* < 0$ is defined in (36).

Solitons. The model (1) admits soliton-type solutions under the conditions [17, 18]:

$$p \in (0, 1), \quad 0 < q < r. \tag{62}$$

When $A < 0$, the first one implies the assumption $\alpha > 0$ again. Next, proceeding in a similar way as we did in Section 4, we obtain the equalities (54). Thus, the inequality $q > 0$ is verified automatically for $A < 0$, whereas $q < r$ implies the following:

$$(c_3 - \alpha^2 r c_1)A > \gamma_\alpha(1 - g_*^r). \tag{63}$$

Obviously, Eq.(63) can be realized under the condition (59) only. Taking into account that $g_* \to 0$ as $q \to r$, we obtain the existence condition for solitons

$$A < A^*. \tag{64}$$

Twins. For $A < 0$, the second equality in (56) implies the same assumption $\alpha > 0$. Thus the existence condition yields now:

$$(c_3 - 2\alpha^2 c_1)A < \gamma_\alpha(1 - g_*^r). \tag{65}$$

6. Conclusion

Let us summarize all the results of the previous sections and [18] to demonstrate the existence of one or the other type of traveling waves in dependence on the wave amplitude. We fix a set of structural constants α, $c_0, \ldots,$ c_3, γ, and very the amplitude $A \in \mathbb{R}^1$. There are several distinct possibilities.

1. Let $\alpha > 0$ and $\gamma_\alpha > 0$. Then there appears a sufficiently complicated system of restrictions which is depicted in **Figure 3**- **Figure 5** for solitons, peakons and cuspons.

As for twins, the formulas (55) and (65) imply the conclusion:

$$\text{if} \quad c_3 < 2\alpha^2 c_1, \quad \text{then} \quad A \in (a^*, 0), \tag{66}$$

$$\text{if} \quad c_3 = 2\alpha^2 c_1, \quad \text{then} \quad A \in \mathbb{R}^1_-, \tag{67}$$

$$\text{if} \quad c_3 > 2\alpha^2 c_1, \quad \text{then} \quad A \in \mathbb{R}^1_- \bigcup \{a^*, \infty\}, \tag{68}$$

where $a^* = \gamma_\alpha(1 - g_*^{r)}/(c_3 - 2\alpha^2 c_1)$. Note that

$a^* > A^*$ in the cases (66) and (68). So, twins belong to the cuspons zone always.

2. Let $\alpha > 0$ and $\gamma_\alpha = 0$. Then for all $A \neq 0$ the traveling waves are:

$$\text{solitons if} \quad c_3 < \alpha^2 r c_1, \tag{69}$$
$$\text{peakons if} \quad c_3 = \alpha^2 r c_1, \tag{70}$$
$$\text{cuspons if} \quad c_3 > \alpha^2 r c_1, \tag{71}$$
$$\text{twins if} \quad c_3 > 2\alpha^2 c_1. \tag{72}$$

3. Let $\alpha = 0$. Then the waves are:

solitons with the amplitudes

$$A \in (0, A^*), \;\; A^* \stackrel{\text{def}}{=} \gamma/c_3, \tag{73}$$

peakon with the amplitude A^* and the velocity $V = c_0 + rc_1\gamma/c_3$,

cuspons with the amplitude A^* and the velocities $V > c_0 + rc_1\gamma/c_3$,

twins with the amplitudes $A > 0$ and the velocities V defined in (56).

For $A < 0$ and $\alpha = 0$ traveling waves don't exist.

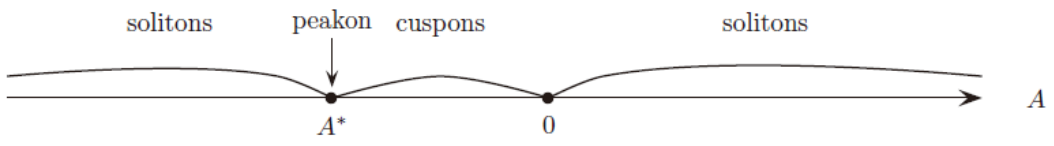

Figure 3: The wave-kind dependence on the amplitude. The case $\alpha > 0$ and $\gamma_\alpha > 0$ when $c_3 < \alpha^2 r c_1$

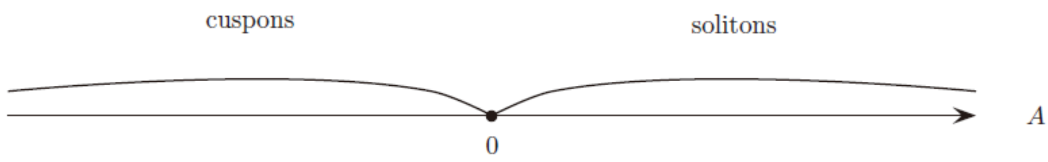

Figure 4: The wave-kind dependence on the amplitude. The case $\alpha > 0$ and $\gamma_\alpha > 0$ when $c_3 = \alpha^2 r c_1$

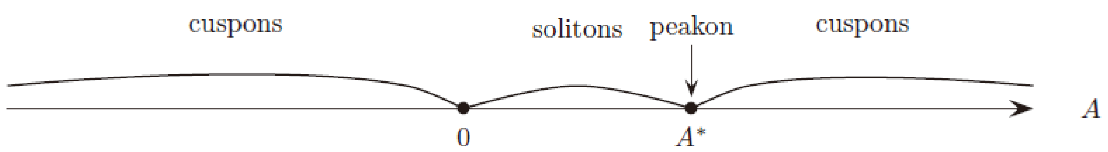

Figure 5: The wave-kind dependence on the amplitude. The case $\alpha > 0$ and $\gamma_\alpha > 0$ when $c_3 > \alpha^2 r c_1$

Remark 3. For solitons with positive amplitudes the condition $q > 0$ implies formally the inequality

$$A > \gamma_\alpha (1 - g_*^r)/c_3. \qquad (74)$$

However, $g_* \to 1$ when $q \to 0$, so that we change (74) to the restriction $A > 0$.

Remark 4. For the standard Camassa-Holm equation (2) $\alpha^2 rc_1 = c_3$. There- fore, if $c_0 > 0$, then $\gamma_\alpha > 0$ and we have the situation depicted in the **Figure 4**. If $c_0 = 0$, then $\gamma_\alpha = 0$ and we have the case (70). The same is true for the Degasperis- Procesi equation. Note also that the CH and DP equations don't admit twins-type solutions.

Remark 5. Recall that we consider Eq.(1) under the assumptions (8) and presuppose that waves vanish at infinity. If $\lim_{x\to\pm\infty} u \neq 0$, then the space of traveling wave solutions is much more rich, see e.g.[9,10,16].

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