

Original research article

A model for polyatomic gases with hyperbolicity and H-theorem satisfied up to whatever order

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Abstract: A relativistic model for polyatomic gases with an arbitrary but fixed number of moments is well-known in the literature. The model's balance equations have symmetric hyperbolic form in their left-hand sides because the tensors that derivated with respect to x^α are gradients of a 4-potential. Here, the symmetric form and a 4-potential are obtained for their right-hand side also, i.e., the production terms. Moreover, this will allow us to prove the H-theorem up to whatever order, while in other articles present in the literature, this result was achieved only up to the second order concerning equilibrium. These obtained results can be derived by either following the Eckart approach or the Landau-Lifshitz one.

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1. Introduction

The most diffused classical approaches to model a dissipative relativistic gas are those of Eckart^[1] and Landau-Lifshitz^[2]. In the study of Carrisi and Pennisi^[3], it has been proved that the two approaches are equivalent; they differ only for a different definition of the deviations from equilibrium and consequent expansion performed to obtain a linear model. We will prove here that, if these expansions would be done up to whatever order, the two approaches would give the same result. These expansions are used to do a transformation of the independent variables from apparently anonymous Lagrange Multipliers to variables which have an immediate physical meaning. However this is just a stylistic issue, as in elementary geometry the algebraic expression of a surface isn't more significant of its parametric expression which uses "anonymous" parameters. So, before to convert the Lagrange Multipliers to the other variables, the two approaches are the same. The Landau-Lifshitz approach has the advantage to obtain immediately a zero production of mass and of momentum-energy; here we will see that it satisfies the H-theorem up to whatever order. We will see that the same result can be obtained also with the Eckart approach. Regarding the hyperbolicity, it has already been proved that it holds up to whatever order if the Lagrange Multipliers are used as independent variables^[4, 5]. Whatever restriction on the so called hyperbolicity region is due to the low degree of approximation which is used for the above unnecessary change of variables. As a confirmation of this fact we see that Profs. Brini and Ruggeri implemented the articles^[6, 7] by considering a better approximation (The second order one) in^[8] and obtained a bigger hyperbolicity zone. This zone would cover all the set of possible values of independent variables if no approximation was used; this is not possible for calculations difficulties, but we cannot expect nature to bow to our mathematical difficulties. Other doubts about the validity of Extended Thermodynamics with many moments arose following Struchtrup's articles^[9, 10] and similar, which studied a transition to Ordinary Thermodynamics and found some weak points. But Extended Thermodynamics arose exactly to overcome the problems of Ordinary Thermodynamics, such as the parabolic equations and the propagation waves with infinite speeds; so it makes no sense to test a better theory through the worse theory that she passed. Moreover, Struchtrup too uses approximations around equilibrium and it makes no sense to increase the number of moments without increasing the order of approximations

around equilibrium. Finally, the methods used for the transition to Ordinary Thermodynamic are mathematically well defined, but there is nothing physical which ensures that the result of these methods really lead to Ordinary Thermodynamics; this doubt is reinforced by the fact the results for bulk viscosity, heat conductivity and shear viscosity are different depending on whether the Chapman-Enskog Method or the Maxwellian Iteration are used (in this latter case it even depends on the number N of moments being used). This is proved by Demontis and Pennisi in^[11]. So we don't take into account these non-existing weakness in the sequel.

The starting point to obtain the balance equations is the Boltzmann–Chernikov equation

$$p^\alpha \partial_\alpha f = Q \quad (1)$$

where f is the distribution function and Q the production term. Arima, Carrisi, Pennisi and Ruggeri found and used the expression of f in^{[12], [5], [13]} and reads

$$f = e^{-1 - \frac{\chi}{k_B}}, \quad \chi = \sum_{n=0}^N \frac{1}{m^{n-1}} \lambda_{\alpha_1 \alpha_2 \dots \alpha_n} p^{\alpha_1} p^{\alpha_2} \dots p^{\alpha_n} \left(1 + \frac{\mathcal{I}}{m c^2} \right)^n \quad (2)$$

where k_B is the Boltzmann constant, c is the speed of light, m is the particle mass, N denotes the number of moments which is used, $\lambda_{\alpha_1 \alpha_2 \dots \alpha_n}$ are the Lagrange multipliers, p^α is the 4-momentum of a particle (satisfying the relation $p^\alpha p_\alpha = m^2 c^2$), \mathcal{I} is the internal energy of a particle due to its internal modes (rotations and vibrations).

Regarding the production term Q , if we adopt the Landau-Lifschiz approach, it has the form

$$Q = -\frac{1}{c^2 \tau} U_{L\alpha} p^\alpha (f - f_{eq.}) \quad (3)$$

where τ is a relaxation time and $U_{L\alpha}$ is the Landau-Lifshitz 4-velocity; its physical meaning has been explained by De Groot, Rezzolla, van Leeuwen, van Weert and Zanotti in^[15, 16]. We will see here that it generates an expression valid also for the Eckart approach. In particular, we will see that, with the Landau-Lifschiz approach, the balance equations take the form

$$\partial_\alpha \frac{\partial h'^\alpha}{\partial \lambda} = 0, \quad \partial_\alpha \frac{\partial h'^\alpha}{\partial \lambda_\beta} = 0, \quad \partial_\alpha \frac{\partial h'^\alpha}{\partial \lambda_{\beta_1 \dots \beta_n}} = \frac{U_\alpha}{c \tau} \frac{\partial Q^\alpha}{\partial \lambda_{\beta_1 \dots \beta_n}} \quad \text{for } n \geq 2 \quad (4)$$

where the gradient of a 4-potential is present both in the left hand sides (here the 4-potential is h'^α) than in the right hand sides (here the 4-potential is Q^α). A less elegant form, but with the same properties, will be obtained by following the Eckart approach and it reads:

$$\partial_\alpha \frac{\partial h'^\alpha}{\partial \lambda_{\beta_1 \dots \beta_n}} = \frac{U_\alpha}{c \tau} \left(\frac{\partial \tilde{Q}^\alpha}{\partial \lambda} \frac{\partial g}{\partial \lambda_{\beta_1 \dots \beta_n}} + \frac{\partial \tilde{Q}^\alpha}{\partial \lambda_\mu} \frac{\partial g_\mu}{\partial \lambda_{\beta_1 \dots \beta_n}} + \frac{\partial \tilde{Q}^\alpha}{\partial \lambda_{\beta_1 \dots \beta_n}} \right) \quad \text{for } n \geq 2 \quad (5)$$

where the 4-potential in the right hand side is \tilde{Q}^α while g and g_μ are known functions which will be presented below. The plan of this article is the following: In the next section we will see the expressions at equilibrium which is the same for both approaches. In section 3 we will see the balance equations outside equilibrium by separating the two approaches into 2 subsections; the above expressions (4) and (5) will be found and the H-theorem will be proved for them up to whatever order with respect equilibrium (obviously, refraining from making approximations).

In section 4 a transformation will be found which allows to obtain the variables of the Landau-Lifschiz approach in terms of those in Eckart approach. Obviously, the same transformation can be done in the inverse direction but we will refrain to do it for the sake of brevity. Approximations will be used, up to first order with respect to equilibrium, only in its subsection 1 to see how previously result in literature can be recovered from the present one. In its subsection 2, approximations will be used up to second order only to show that the results previously obtained in literature don't hold at any order.

2. The model for relativistic polyatomic gas at equilibrium

At equilibrium the Landau-Litschiz and the Eckart approach give the same expressions. In particular we have the balance equations

$$\partial_\alpha V_E^\alpha = 0, \quad \partial_\alpha T_E^{\alpha\beta} = 0, \quad \text{where} \quad V_E^\alpha = \rho U^\alpha, \quad T_E^{\alpha\beta} = \frac{e}{c^2} U^\alpha U^\beta + p h^{\alpha\beta}$$

$$U_\alpha U^\alpha = c^2, \quad h^{\alpha\beta} = -g^{\alpha\beta} + \frac{U^\alpha U^\beta}{c^2} \quad (\text{The projector into the subspace orthogonal to } U^\alpha)$$

Here ρ is the mass density, U^α is the 4-velocity, e the energy density and p the pressure. If we want to find an approach which holds for whatever type of gas, we can introduce the 4-potential

$$h'^\alpha = -4\pi m^3 c^5 h_0 (\lambda^E, \gamma) \frac{\lambda_E^\alpha}{\gamma}, \quad \text{where} \quad \gamma = \frac{m c}{k_B} \sqrt{\lambda_E^\alpha \lambda_\alpha^E} \rightarrow \lambda_E^\alpha \lambda_\alpha^E = \left(\frac{k_B \gamma}{m c}\right)^2$$

(The constant coefficients have been introduced for an easier comparison with expressions previously known in literature). It follows

$$V_E^\alpha = \frac{\partial h'^\alpha}{\partial \lambda_E} = -4\pi m^3 c^5 \frac{\partial h_0}{\partial \lambda_E} \frac{\lambda_E^\alpha}{\gamma}, \quad T_E^{\alpha\beta} = \frac{\partial h'^\alpha}{\partial \lambda_E^\beta} = -4\pi m^3 c^5 \left(\frac{h_0}{\gamma} h^{\alpha\beta} + \frac{\partial h_0}{\partial \gamma} \frac{\lambda_E^\alpha \lambda_E^\beta}{\lambda_E^\mu \lambda_\mu^E} \right)$$

As consequence of these results we have

$$\rho = -4 k_B \pi m^2 c^3 \frac{\partial h_0}{\partial \lambda_E}, \quad U^\alpha = \frac{m c^2}{k_B} \frac{\lambda_E^\alpha}{\gamma}, \quad p = 4\pi m^3 c^5 \frac{h_0}{\gamma}, \quad e = -4\pi m^3 c^5 \frac{\partial h_0}{\partial \gamma} \quad (6)$$

If we know the constitutive function $e = e(\rho, \gamma)$, then (6)₄ becomes

$$e \left(-4 k_B \pi m^2 c^3 \frac{\partial h_0}{\partial \lambda_E}, \gamma \right) = -4\pi m^3 c^5 \frac{\partial h_0}{\partial \gamma}$$

which is a differential equation from which to deduce h_0 . For example, if $\frac{e}{\rho c^2} = \epsilon(\gamma)$ and we define $\eta(\gamma)$ from $\eta'(\gamma) = \epsilon(\gamma)$, this equation becomes $\frac{\partial h_0}{\partial \gamma} - \frac{k_B}{m} \eta' \frac{\partial h_0}{\partial \lambda_E} = 0$. By considering h_0 a composite function of $H_0(X, Y)$ and of $X = \lambda^E + \frac{k_B}{m} \eta(\gamma)$, $Y = \gamma$, this differential equation becomes $\frac{\partial H_0}{\partial Y} = 0$ so that the general solution is $h_0 = H_0 \left(\lambda^E + \frac{k_B}{m} \eta(\gamma) \right)$ for whatever single variable function H_0 . A further subcase is that when

$$\epsilon = \frac{e}{\rho c^2} = \frac{\int_0^{+\infty} J_{22}(\gamma^*) \left(1 + \frac{\mathcal{I}}{m c^2}\right) \varphi(\mathcal{I}) d\mathcal{I}}{\int_0^{+\infty} J_{21}(\gamma^*) \varphi(\mathcal{I}) d\mathcal{I}} \rightarrow \eta(\gamma) = -\ln \int_0^{+\infty} J_{21}(\gamma^*) \varphi(\mathcal{I}) d\mathcal{I}$$

$$\text{Here } J_{m,n}(\gamma) = \int_0^{+\infty} e^{-\gamma \cosh s} \sinh^m s \cosh^n s ds$$

which can be related to the modified Bessel functions $K_n(\gamma) = \int_0^{+\infty} e^{-\gamma \cosh s} \cosh(ns) ds$; moreover $\gamma^* = \gamma \left(1 + \frac{\mathcal{I}}{m c^2}\right)$.

By taking $H_0(X) = e^{-1 - \frac{mX}{k_B}}$ we find $h_0 = e^{-1 - \frac{m\lambda^E}{k_B}} \int_0^{+\infty} J_{21}(\gamma^*) \varphi(\mathcal{I}) d\mathcal{I}$ and (6)_{1,3} become

$$\rho = 4\pi m^3 c^3 e^{-1 - \frac{m\lambda^E}{k_B}} \int_0^{+\infty} J_{21}(\gamma^*) \varphi(\mathcal{I}) d\mathcal{I}, \quad \frac{p}{\rho} = \frac{c^2}{\gamma}$$

as in Equations (26) and (41) of^[12]. For the sake of simplicity, we will use in the sequel this simpler expression.

3. The dissipative case and the production term

3.1. The Eckart approach

The balance equations for this case have been found by Arima, Carrisi, Pennisi, Ruggeri in^[13], by multiplying Equation (1) with $\frac{c}{m^{n-1}} p^{\alpha_1} p^{\alpha_2} \dots p^{\alpha_n} \left(1 + \frac{\mathcal{I}}{m c^2}\right)^n$ and integrating in $d\mathcal{I} d\vec{P}$, the balance equations for this case have been found

$$\partial_\alpha A^{\alpha\alpha_1 \dots \alpha_n} = I^{\alpha_1 \dots \alpha_n}, \quad \text{for } n = 0, 1, \dots, N \tag{7}$$

where

$$\begin{aligned} A^{\alpha\alpha_1 \dots \alpha_n} &= \frac{\partial h'^\alpha}{\partial \lambda_{\alpha_1 \dots \alpha_n}}, \quad h'^\alpha = -k_B c \int_{\mathbb{R}^3} \int_0^{+\infty} f p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} \\ I^{\alpha_1 \dots \alpha_n} &= \frac{c}{m^{n-1}} \int_{\mathbb{R}^3} \int_0^{+\infty} Q p^{\alpha_1} p^{\alpha_2} \dots p^{\alpha_n} \left(1 + \frac{\mathcal{I}}{m c^2}\right)^n \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} \end{aligned} \tag{8}$$

In particular, (8)₁ for $n = 0, 1$ are respectively the mass conservation law and that of energy-momentum; obviously, we have $A^\alpha = V^\alpha$, $A^{\alpha\alpha_1} = T^\alpha$. Moreover, (8)₂ for $n = 0, 1$ must give $I = 0$, $I^\alpha = 0$. It is easy to see that the left hand side of the balance Equation (7) takes the elegant expression $\partial_\alpha \frac{\partial h'^\alpha}{\partial \lambda_{\alpha_1 \dots \alpha_n}}$. We will show now that also for the right hand side we can obtain a similar expression $I^{\alpha_1 \dots \alpha_n} = \frac{U_\alpha}{c\tau} \frac{\partial Q^\alpha}{\partial \lambda_{\beta_1 \dots \beta_n}}$ with Q^α which will be found later and τ a relaxation time. To this end we need to know the production term Q . Its expression proposed in^[14] was an approximated one and, in fact, it gave an entropy production σ which was non negative only up to second order with respect to equilibrium. To obtain $\sigma \geq 0$ up to whatever order, we note firstly that, for every expressions of the functions $g(\lambda_{\alpha_1 \alpha_2 \dots \alpha_n})$, $g_\mu(\lambda_{\alpha_1 \alpha_2 \dots \alpha_n})$ with $n \geq 2$, we can define λ^E, λ_μ^E from

$$\lambda = \lambda^E + g(\lambda_{\alpha_1 \alpha_2 \dots \alpha_n}), \quad \lambda_\mu = \lambda_\mu^E + g_\mu(\lambda_{\alpha_1 \alpha_2 \dots \alpha_n}) \tag{9}$$

This can be also considered as a change of independent variables from $\lambda, \lambda_\mu, \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}$ to $\lambda^E, \lambda_\mu^E, \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}$ for $n \geq 2$. We choose here $g(\lambda_{\alpha_1 \alpha_2 \dots \alpha_n}), g_\mu(\lambda_{\alpha_1 \alpha_2 \dots \alpha_n})$ the solution of the conditions $g^E = 0$,

$g_\mu^E = 0$ and of

$$\begin{aligned}
 & U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} - 1 \right) p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = 0 \\
 & U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} - 1 \right) p^\alpha p^\beta \left(1 + \frac{\mathcal{I}}{m c^2} \right) \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = 0 \\
 & \text{i.e., } U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} f_E e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = \frac{U_\alpha V^\alpha}{m c} = \frac{\rho c}{m} \\
 & U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} f_E e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = \frac{U_\alpha}{c} T_E^{\alpha\beta} = \frac{e}{c} U^\beta
 \end{aligned} \tag{10}$$

with $\Delta\chi = \sum_{n=2}^N \frac{1}{m^{n-1}} \lambda_{\alpha_1 \alpha_2 \dots \alpha_n} p^{\alpha_1} p^{\alpha_2} \dots p^{\alpha_n} \left(1 + \frac{\mathcal{I}}{m c^2} \right)^n$. We will prove in the appendix that this problem gives one and only one solution. After that, we propose for Q the following expression

$$Q = - \frac{U_\alpha p^\alpha}{c^2 \tau} f_E \left(e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2})]} e^{\frac{-\Delta\chi}{k_B}} - 1 \right) \tag{11}$$

We note that, with this expression of Q we have $Q^E = 0$ and $(8)_3$ gives $I = 0, I^\alpha = 0$ automatically (Thanks to (10)) so that the conservation laws of mass and of momentum-energy are satisfied up to whatever order; moreover, we prove now the following theorem:

Theorem 1. "The production terms $I^{\alpha_1 \dots \alpha_n}$ of the balance equations in Equation $(8)_2$ can be expressed as:

$$\begin{aligned}
 I^{\alpha_1 \dots \alpha_n} &= \frac{U_\alpha}{c \tau} \frac{\partial Q^\alpha}{\partial \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}}, \quad \text{with } Q^\alpha = g \frac{\rho}{c} U^\alpha + \frac{e}{c^3} (U^\mu g_\mu) U^\alpha + \\
 &+ k_B \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} + \frac{\Delta\chi}{k_B} \right) p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} ".
 \end{aligned} \tag{12}$$

(Note that ρ, U^α and the energy e depend on λ^E, λ_μ^E and not on $\lambda_{\alpha_1 \alpha_2 \dots \alpha_n}$ for $n \geq 2$).

Proof 1. To prove the previous result, we can calculate

$$\begin{aligned}
 \frac{\partial Q^\alpha}{\partial \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}} &= \frac{\rho}{c} U^\alpha \frac{\partial g}{\partial \lambda_{\alpha_1 \dots \alpha_n}} + \frac{e}{c^3} U^\mu U^\alpha \frac{\partial g_\mu}{\partial \lambda_{\alpha_1 \dots \alpha_n}} - \\
 &\int_{\mathbb{R}^3} \int_0^{+\infty} f_E e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} \left[m \frac{\partial g}{\partial \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}} + p^\mu \frac{\partial g_\mu}{\partial \lambda_{\alpha_1 \dots \alpha_n}} \left(1 + \frac{\mathcal{I}}{m c^2} \right) \right] p^\alpha \cdot \\
 &\cdot \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} - \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} - 1 \right) \frac{\partial \Delta\chi}{\partial \lambda_{\alpha_1 \dots \alpha_n}} p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P}
 \end{aligned}$$

by contracting with $\frac{U_\alpha}{c \tau}$, and taking into account that $U^\alpha U_\alpha = c^2$ we obtain

$$\begin{aligned}
 \frac{U_\alpha}{c \tau} \frac{\partial Q^\alpha}{\partial \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}} &= \frac{\rho}{\tau} \frac{\partial g}{\partial \lambda_{\alpha_1 \dots \alpha_n}} + \frac{e}{c^2 \tau} U^\mu \frac{\partial g_\mu}{\partial \lambda_{\alpha_1 \dots \alpha_n}} - \\
 \frac{U_\alpha}{c \tau} \int_{\mathbb{R}^3} \int_0^{+\infty} f_E e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} &\left[m \frac{\partial g}{\partial \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}} + p^\mu \frac{\partial g_\mu}{\partial \lambda_{\alpha_1 \dots \alpha_n}} \left(1 + \frac{\mathcal{I}}{m c^2} \right) \right] p^\alpha \cdot \\
 \cdot \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} - \frac{U_\alpha}{c \tau} \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} - 1 \right) &\frac{\partial \Delta\chi}{\partial \lambda_{\alpha_1 \dots \alpha_n}} p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P}
 \end{aligned}$$

which can be reordered as

$$\begin{aligned} \frac{U_\alpha}{c\tau} \frac{\partial Q^\alpha}{\partial \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}} &= \\ &= \frac{1}{c^2 \tau} \frac{\partial g}{\partial \lambda_{\alpha_1 \dots \alpha_n}} \left(\rho c^2 - U_\alpha m c \int_{\mathbb{R}^3} \int_0^{+\infty} f_E e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta \chi]} p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} \right) + \\ &+ \frac{1}{c^2 \tau} \frac{\partial g_\mu}{\partial \lambda_{\alpha_1 \dots \alpha_n}} \left(e U^\mu - U_\alpha c \int_{\mathbb{R}^3} \int_0^{+\infty} f_E e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta \chi]} p^\mu \right) \cdot \\ &\cdot \left(1 + \frac{\mathcal{I}}{m c^2} \right) p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} \Big) + c \int_{\mathbb{R}^3} \int_0^{+\infty} Q \frac{\partial \Delta \chi}{\partial \lambda_{\alpha_1 \dots \alpha_n}} \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} \end{aligned}$$

and here the coefficients of $\frac{\partial g}{\partial \lambda_{\alpha_1 \dots \alpha_n}}$ and of $\frac{\partial g_\mu}{\partial \lambda_{\alpha_1 \dots \alpha_n}}$ are identically zero for (10)_{3,4}. Moreover it has been used $f_E \left(e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2})]} e^{\frac{-\Delta \chi}{k_B}} - 1 \right) = -\frac{c^2 \tau}{U_\alpha p^\alpha} Q$ which follows from (11). Finally, $\Delta \chi$ is given after eqs. (10) and it yields $\frac{\partial \Delta \chi}{\partial \lambda_{\alpha_1 \dots \alpha_n}} = \frac{1}{m^{n-1}} p^{\alpha_1} p^{\alpha_2} \dots p^{\alpha_n} \left(1 + \frac{\mathcal{I}}{m c^2} \right)^n$, so that the above relation becomes

$$\frac{U_\alpha}{c\tau} \frac{\partial Q^\alpha}{\partial \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}} = c \int_{\mathbb{R}^3} \int_0^{+\infty} Q \frac{1}{m^{n-1}} p^{\alpha_1} p^{\alpha_2} \dots p^{\alpha_n} \left(1 + \frac{\mathcal{I}}{m c^2} \right)^n \varphi(\mathcal{I}) d\mathcal{I} d\vec{P}$$

Thanks to the definition (8)₂, this gives the result $\frac{U_\alpha}{c\tau} \frac{\partial Q^\alpha}{\partial \lambda_{\alpha_1 \alpha_2 \dots \alpha_n}} = I^{\alpha_1 \dots \alpha_n}$. In this way the proof of (12) and of the theorem is completed. \square

Hence the balance equations take the elegant form (4) where the gradient form with respect to $\lambda_{\beta_1 \dots \beta_n}$ is present both in the left and in the right hand side. Obviously, there is in (4) the drawback that the left hand sides uses the variables $\lambda, \lambda_\alpha, \lambda_{\alpha_1 \dots \alpha_n}$ while the right hand sides uses the variables $\lambda^E, \lambda_\alpha^E, \lambda_{\alpha_1 \dots \alpha_n}$; however we can express Q^α in terms of the old variables by substituting in it the inverse transformation of (9). In this way Q^α is the composite function of $\tilde{Q}^\alpha(\lambda, \lambda_\alpha, \lambda_{\alpha_1 \dots \alpha_n})$ and of $\lambda = \lambda^E + g, \lambda_\alpha = \lambda_\alpha^E + g_\alpha, \lambda_{\alpha_1 \dots \alpha_n}$ and (4)₃ becomes (5) where the gradient form appears also in the right hand side even if through 3 terms.

There remains to prove in this section the

H-Theorem 1. "The entropy production $\sigma = \sum_{n=2}^N I^{\beta_1 \dots \beta_n} \lambda_{\beta_1 \dots \beta_n}$ is non negative and is zero only at equilibrium".

To prove it we calculate

$$\begin{aligned} \sigma &= -\frac{U_\alpha}{c\tau} \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2})]} e^{\frac{-\Delta \chi}{k_B}} - 1 \right) \Delta \chi p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = \\ &= \frac{k_B U_\alpha}{c\tau} \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(-e^{\frac{-1}{k_B} [m g + p^\mu g_\mu (1 + \frac{\mathcal{I}}{m c^2})]} e^{\frac{-\Delta \chi}{k_B}} + 1 \right) p^\alpha + \\ &+ \frac{1}{k_B} \left[\Delta \chi + \underline{m g + g_\mu p^\mu \left(1 + \frac{\mathcal{I}}{m c^2} \right)} \right] \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} \end{aligned} \tag{13}$$

Here the underlined terms give a zero contribute thanks to (10); they have been included for convenience of calculations. In fact, in this way the function to be integrated has the form

$$F(x) = (1 - e^{-x}) x \quad \text{with} \quad x = \frac{1}{k_B} \left[\Delta \chi + m g + g_\mu p^\mu \left(1 + \frac{\mathcal{I}}{m c^2} \right) \right]$$

and we have

$$F'(x) = 1 + e^{-x}(x - 1), F''(x) = e^{-x}(-x + 2), F'(0) = 0, \lim_{x \rightarrow +\infty} F'(x) = 1$$

From these calculations we note that for $x < 2$ the function $F'(x)$ is increasing so that it can have only a root which is $x = 0$; for $x \geq 2$ it is a decreasing function and goes from $1 + e^{-2}$ to 1 so that $F'(x) > 0$ for $x \geq 2$. It follows that the function $F(x)$ is decreasing for $x < 0$ and increasing for $x > 0$; therefore it has a minimum value in $x = 0$. Since $g(0) = 0$, it follows that $g(0) > 0 \forall x \neq 0$, as we wanted to prove.

In order to compare the present results with those of Pennisi and Ruggeri in^[14], we conclude this subsection by seeing its implication on the linear expressions. At first order with respect to equilibrium, Equations (10) become

$$\begin{aligned} \rho c^2 g^{(1)} + e g_\mu^{(1)} U^\mu + U_\alpha \sum_{n=2}^N A_E^{\alpha\alpha_1 \dots \alpha_n} \lambda_{\alpha_1 \dots \alpha_n} &= 0, \\ e U^\beta g^{(1)} + U_\alpha A_E^{\mu\alpha\beta} g_\mu^{(1)} + U_\alpha \sum_{n=2}^N A_E^{\alpha\beta\alpha_1 \dots \alpha_n} \lambda_{\alpha_1 \dots \alpha_n} &= 0 \end{aligned} \tag{14}$$

where $g^{(1)}$ and $g_\mu^{(1)}$ denote the first order terms of g and g_μ respectively. But in the Eckart approach we have also the following conditions:

$$\begin{aligned} V^\alpha - V_E^\alpha &= 0, \quad U_\alpha U_\beta (T^{\alpha\beta} - T_E^{\alpha\beta}) = 0, \\ h_\alpha^\delta U_\beta (T^{\alpha\beta} - T_E^{\alpha\beta}) &= -q^\delta \rightarrow U_\beta (T^{\alpha\beta} - T_E^{\alpha\beta}) = q^\alpha \end{aligned} \tag{15}$$

where (15)₃ isn't a condition but only the definition of the heat flux q^α ; moreover, (15)₄ is a consequence of (15)_{2,3} and of $U_\alpha q^\alpha = 0$. The conditions (15)_{1,2} at first order with respect to equilibrium become

$$\begin{aligned} \left(\frac{\partial V^\alpha}{\partial \lambda} \right)_E (\lambda - \lambda^E)^{(1)} + \left(\frac{\partial V^\alpha}{\partial \lambda_\mu} \right)_E (\lambda_\mu - \lambda_\mu^E)^{(1)} + \sum_{n=2}^N \left(\frac{\partial V^\alpha}{\partial \lambda_{\alpha_1 \dots \alpha_n}} \right)_E \lambda_{\alpha_1 \dots \alpha_n} &= 0, \\ U_\alpha U_\beta \left[\left(\frac{\partial T^{\alpha\beta}}{\partial \lambda} \right)_E (\lambda - \lambda^E)^{(1)} + \left(\frac{\partial T^{\alpha\beta}}{\partial \lambda_\mu} \right)_E (\lambda_\mu - \lambda_\mu^E)^{(1)} + \right. \\ \left. + \sum_{n=2}^N \left(\frac{\partial T^{\alpha\beta}}{\partial \lambda_{\alpha_1 \dots \alpha_n}} \right)_E \lambda_{\alpha_1 \dots \alpha_n} \right] &= 0 \end{aligned} \tag{16}$$

But $V^\alpha = A^\alpha, T^{\alpha\beta} = A^{\alpha\beta}$ and we can use (10)_{1,2} with f given by (2) which imply

$$\begin{aligned} \frac{\partial V^\alpha}{\partial \lambda} &= -\frac{m}{k_B} V^\alpha, \quad \frac{\partial V^\alpha}{\partial \lambda_\mu} = -\frac{m}{k_B} T^{\alpha\mu}, \quad \frac{\partial V^\alpha}{\partial \lambda_{\alpha_1 \dots \alpha_n}} = -\frac{m}{k_B} A^{\alpha\alpha_1 \dots \alpha_n} \\ \frac{\partial T^{\alpha\beta}}{\partial \lambda} &= -\frac{m}{k_B} T^{\alpha\beta}, \quad \frac{\partial T^{\alpha\beta}}{\partial \lambda_\mu} = -\frac{m}{k_B} A^{\alpha\beta\mu}, \quad \frac{\partial T^{\alpha\beta}}{\partial \lambda_{\alpha_1 \dots \alpha_n}} = -\frac{m}{k_B} A^{\alpha\beta\alpha_1 \dots \alpha_n} \end{aligned}$$

Thanks to these relations, (16) becomes

$$\begin{aligned} \rho U^\alpha (\lambda - \lambda^E)^{(1)} + \left(\frac{e}{c^2} U^\alpha U^\mu + p h^{\alpha\mu}\right) (\lambda_\mu - \lambda_\mu^E)^{(1)} + \sum_{n=2}^N A_E^{\alpha\alpha_1 \dots \alpha_n} \lambda_{\alpha_1 \dots \alpha_n} &= 0 \\ e U^\alpha (\lambda - \lambda^E)^{(1)} + U_\beta A_E^{\alpha\beta\mu} (\lambda_\mu - \lambda_\mu^E)^{(1)} + U_\beta \sum_{n=2}^N A_E^{\alpha\beta\alpha_1 \dots \alpha_n} \lambda_{\alpha_1 \dots \alpha_n} &= -\frac{k_B}{m} q^\alpha \end{aligned} \tag{17}$$

We can deduce from these equations $\sum_{n=2}^N A_E^{\alpha\alpha_1 \dots \alpha_n} \lambda_{\alpha_1 \dots \alpha_n}$ and $U_\beta \sum_{n=2}^N A_E^{\alpha\beta\alpha_1 \dots \alpha_n} \lambda_{\alpha_1 \dots \alpha_n}$; by substituting them in (14) this equation becomes

$$\begin{aligned} \rho c^2 \left[g^{(1)} - (\lambda - \lambda^E)^{(1)} \right] + e \left[g_\mu^{(1)} - (\lambda_\mu - \lambda_\mu^E)^{(1)} \right] U^\mu &= 0, \\ e U^\beta \left[g^{(1)} - (\lambda - \lambda^E)^{(1)} \right] + U_\alpha A_E^{\mu\alpha\beta} \left[g_\mu^{(1)} - (\lambda_\mu - \lambda_\mu^E)^{(1)} \right] - \frac{k_B}{m} q^\beta &= 0 \end{aligned} \tag{18}$$

Now Equation (18)₁, (18)₂ contracted with U_β and (18)₂ contracted with h_β^δ give

$$\begin{aligned} g^{(1)} &= (\lambda - \lambda^E)^{(1)}, \quad g_\mu^{(1)} U^\mu = (\lambda_\mu - \lambda_\mu^E)^{(1)} U^\mu \\ h^{\mu\delta} \left[g_\mu^{(1)} - (\lambda_\mu - \lambda_\mu^E)^{(1)} \right] &= \frac{3}{\rho c^4 \theta_{1,2}} \frac{k_B}{m} q^\delta \end{aligned} \tag{19}$$

where Equation (14), found Arima, Carrisi, Pennisi and Ruggeri in^[13] has been used. Now (11) can be written as

$$Q = -\frac{U_\alpha p^\alpha}{c^2 \tau} f_E \left[\frac{f}{f_E} - 1 + \frac{f}{f_E} \left(e^{\frac{-1}{k_B} \left[m (g - \lambda + \lambda^E) + p^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E) \right] \left(1 + \frac{\mathcal{I}}{m c^2} \right)} - 1 \right) \right]$$

Here we can linearize the term $\frac{f}{f_E} \left(e^{\frac{-1}{k_B} \left[m (g - \lambda + \lambda^E) + p^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E) \right] \left(1 + \frac{\mathcal{I}}{m c^2} \right)} - 1 \right)$, i.e., substitute it with

$$\begin{aligned} &\left(\frac{f}{f_E} \right)^{(0)} \left(e^{\frac{-1}{k_B} \left[m (g - \lambda + \lambda^E) + p^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E) \right] \left(1 + \frac{\mathcal{I}}{m c^2} \right)} - 1 \right)^{(1)} + \\ &+ \left(\frac{f}{f_E} \right)^{(1)} \left(e^{\frac{-1}{k_B} \left[m (g - \lambda + \lambda^E) + p^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E) \right] \left(1 + \frac{\mathcal{I}}{m c^2} \right)} - 1 \right)^{(0)} = \\ &= \frac{-1}{k_B} \left[m (g - \lambda + \lambda^E)^{(1)} + p^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(1)} \right] \left(1 + \frac{\mathcal{I}}{m c^2} \right) \end{aligned}$$

Moreover, we use the result (19) and find

$$Q = -\frac{U_\alpha p^\alpha}{c^2 \tau} f_E \left[\frac{f}{f_E} - 1 + p^\mu q_\mu \frac{3}{m \rho c^4 \theta_{1,2}} \left(1 + \frac{\mathcal{I}}{m c^2} \right) \right]$$

which is the expression that Arima, Carrisi, Pennisi and Ruggeri have found in Equation (43) of^[13].

3.2. The Landau-Lifschiz approach

We note that all the considerations in the previous subsection, up to its Equation (4), can be repeated also in the present approach and Equations (10) simply means that

$$U_{\alpha} (V^{\alpha} - V_{eq}^{\alpha}) = 0, \quad U_{\alpha} (T^{\alpha\beta} - T_{eq}^{\alpha\beta}) = 0 \quad (20)$$

because by using (9), we have

$$f_E \left(e^{\frac{-1}{k_B} \left[m g + p^{\mu} g_{\mu} \left(1 + \frac{\tau}{m c^2} \right) + \Delta \chi \right]} - 1 \right) = f - f_E$$

Moreover, the expression (11) of the production term can be written simply as

$$Q = \frac{-U_{\alpha} p^{\alpha}}{c^2 \tau} (f - f_E)$$

From this viewpoint it seems that the definition of deviations from equilibrium, which is present in the Landau-Lifschiz approach, was purposely made to automatically achieve zero mass production and zero momentum-energy production:

$$I = -\frac{U_{L\alpha}}{c^2 \tau} (V^{\alpha} - V_E^{\alpha}) = 0, \quad I^{\beta} = -\frac{U_{L\alpha}}{c^2 \tau} (T^{\alpha\beta} - T_E^{\alpha\beta}) = 0$$

So also in this approach we obtain the Equation (4), where the gradient form with respect to $\lambda_{\beta_1 \dots \beta_n}$ is present both in the left and in the right hand side. Moreover, we don't have here the drawback which was present in (4) which forced us to transform it in the less elegant form (5). It is true that also in the present approach the left hand sides of (4) uses the variables $\lambda, \lambda_{\alpha}, \lambda_{\alpha_1 \dots \alpha_n}$ while the right hand sides uses the variables $\lambda^E, \lambda_{\alpha}^E, \lambda_{\alpha_1 \dots \alpha_n}^E$; but, instead of expressing Q^{α} in terms of the old variables, we can express the left hand sides (i.e., h'^{α}) in terms of the new variables. Since an invertible change of independent variables maintains the hyperbolicity requirement, as long as this change of independent variables is not done in an approximated way, this requirement is here preserved. Also the proof of the H-Theorem which is present in the previous subsection, between Equations (5) and (14) still holds also in the present case.

We see also that the expression of Q proposed by Carrisi and Pennisi in Equation (7) of^[4] (with $a_1 = 0, a_2 = 1/\tau$) for the Eckart approach is the same of the present one (11), except to identify the functions ψ and ψ_{μ} of^[4] respectively with the functions g and g_{μ} which are here present. However, they were introduced in^[4] ad hoc, as a mathematical tool; instead here we have seen that they come from trying to adapt the Eckart approach to the Landau-Lifschiz approach.

4. Determination of q_L^α and $P^{\alpha\beta}$ in terms of the variables in the Eckart approach

Both the approaches are the same at equilibrium so that they give the same values for ρ , U^α , e and p ; in particular they give

$$\rho U^\alpha = \frac{\partial h_E'^\alpha}{\partial \lambda_E}, T_E^{\alpha\beta} = \frac{e}{c^2} U^\alpha U^\beta + p h^{\alpha\beta} = \frac{\partial h_E'^\alpha}{\partial \lambda_E^\beta} \quad \text{with}$$

$$h_E'^\alpha = -k_B c \int_{\mathbb{R}^3} \int_0^{+\infty} f_E p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P}, f_E = e^{-1 - \frac{1}{k_B} [m \lambda^E + \lambda_\mu^E p^\mu (1 + \frac{\mathcal{I}}{m c^2})]}$$

Outside equilibrium, with the Eckart approach we have $f = f^{Ek}$ with

$$f^{Ek} = f_E e^{-\frac{1}{k_B} [m(\lambda - \lambda^E) + (\lambda_\mu - \lambda_\mu^E) p^\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]}$$

$$\Delta\chi = \sum_{n=2}^N \frac{1}{m^{n-1}} \lambda_{\alpha_1 \dots \alpha_n} p^{\alpha_1} \dots p^{\alpha_n} \left(1 + \frac{\mathcal{I}}{m c^2}\right)^n$$

while with the Landau-Lifschiz approach we have $f = f^L$ with

$$f^L = f_E e^{-\frac{1}{k_B} [m g + g_\mu p^\mu (1 + \frac{\mathcal{I}}{m c^2}) + \Delta\chi]} \rightarrow$$

$$\frac{f^L}{f^{Ek}} = e^{-\frac{1}{k_B} [m(g - \lambda + \lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu (1 + \frac{\mathcal{I}}{m c^2})]}, \tag{21}$$

$$\frac{f^L - f_E}{f_E} = \frac{f^{Ek}}{f_E} \left(e^{-\frac{1}{k_B} [m(g - \lambda + \lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu (1 + \frac{\mathcal{I}}{m c^2})]} - 1 \right) + \frac{f^{Ek}}{f_E} - 1$$

Now q_L^α and $P^{\alpha\beta}$ are defined by $q_L^\alpha = V_L^\alpha - \rho U_L^\alpha$ and $P^{\alpha\beta} = T_L^{\alpha\beta} - T_E^{\alpha\beta}$ so that we have the system

$$q_L^\alpha = m c \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \frac{f^L - f_E}{f_E} p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P},$$

$$P^{\alpha\beta} = c \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \frac{f^L - f_E}{f_E} p^\alpha p^\beta \left(1 + \frac{\mathcal{I}}{m c^2}\right) \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} \tag{22}$$

$$U_\alpha q_L^\alpha = 0, \quad U_\alpha P^{\alpha\beta} = 0$$

for the determination of $g - \lambda + \lambda^E$, $g_\mu - \lambda_\mu + \lambda_\mu^E$, q_L^α , $P^{\alpha\beta}$. To this end, from (21)₃ we have that $\frac{f^L - f_E}{f_E}$ at the order zero is zero, while at the orders 1 and 2 is respectively given by

$$\left(\frac{f^L - f_E}{f_E}\right)^{(1)} = \left(\frac{f^{Ek}}{f_E}\right)^{(0)} \left(e^{-\frac{1}{k_B} [m(g - \lambda + \lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu (1 + \frac{\mathcal{I}}{m c^2})]} - 1 \right)^{(1)} +$$

$$+ \left(\frac{f^{Ek}}{f_E}\right)^{(1)} \left(e^{-\frac{1}{k_B} [m(g - \lambda + \lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu (1 + \frac{\mathcal{I}}{m c^2})]} - 1 \right)^{(0)} + \left(\frac{f^{Ek}}{f_E} - 1\right)^{(1)} = \tag{23}$$

$$= -\frac{1}{k_B} \left[m(g - \lambda + \lambda^E)^{(1)} + (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(1)} p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right] + \left(\frac{f^{Ek}}{f_E} - 1\right)^{(1)}$$

$$\begin{aligned} \left(\frac{f^L - f_E}{f_E}\right)^{(2)} &= \left(\frac{f^{Ek}}{f_E}\right)^{(0)} \left(e^{-\frac{1}{k_B} [m(g - \lambda + \lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E)p^\mu (1 + \frac{\mathcal{I}}{m c^2})]} - 1 \right)^{(2)} + \\ &+ \left(\frac{f^{Ek}}{f_E}\right)^{(1)} \left(e^{-\frac{1}{k_B} [m(g - \lambda + \lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E)p^\mu (1 + \frac{\mathcal{I}}{m c^2})]} - 1 \right)^{(1)} + \\ &+ \left(\frac{f^{Ek}}{f_E}\right)^{(2)} \left(e^{-\frac{1}{k_B} [m(g - \lambda + \lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E)p^\mu (1 + \frac{\mathcal{I}}{m c^2})]} - 1 \right)^{(0)} + \left(\frac{f^{Ek}}{f_E} - 1\right)^{(2)} \end{aligned}$$

where the underlined terms are zero.

4.1. The expressions of q_L^α and $P^{\alpha\beta}$ at first order with respect to equilibrium

By using the above passages we see that the system (22) at first order becomes

$$\begin{aligned} (q_L^\alpha)^{(1)} &= m c \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(\frac{f^L - f_E}{f_E}\right)^{(1)} p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = \\ &= -\frac{m}{k_B} (g - \lambda + \lambda^E)^{(1)} \rho U^\alpha - \frac{m}{k_B} (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(1)} T_E^{\alpha\mu} + \underline{(V_{Ek}^\alpha - V_E^\alpha)^{(1)}}, \\ (P^{\alpha\beta})^{(1)} &= c \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(\frac{f^L - f_E}{f_E}\right)^{(1)} p^\alpha p^\beta \left(1 + \frac{\mathcal{I}}{m c^2}\right) \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = \\ &= -\frac{m}{k_B} (g - \lambda + \lambda^E)^{(1)} T_E^{\alpha\beta} - \frac{m}{k_B} (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(1)} A_E^{\alpha\beta\mu} + \underline{(T_{Ek}^{\alpha\beta} - T_E^{\alpha\beta})^{(1)}} \\ U_\alpha (q_L^\alpha)^{(1)} &= 0, \quad U_\alpha (P^{\alpha\beta})^{(1)} = 0 \end{aligned} \tag{24}$$

Here too the underlined term is zero, while

$$\begin{aligned} T_E^{\alpha\beta} &= \frac{e}{c^2} U^\alpha U^\beta + p h^{\alpha\beta}, \quad A_E^{\alpha\beta\mu} = \rho \theta_{0,2} U^\alpha U^\beta U^\mu + \rho c^2 \theta_{1,2} U^{(\alpha} h^{\beta\mu)}, \\ (T_{Ek}^{\alpha\beta} - T_E^{\alpha\beta})^{(1)} &= \pi h^{\alpha\beta} + \frac{2}{c^2} U^{(\alpha} q^{\beta)} + t^{<\alpha\beta>} \end{aligned}$$

as it was proved by Arima, Carrisi, Pennisi and Ruggeri in^[13], in particular from its Equation (14). By contracting Equation (24)₁ with U_α , eq. (24)₂ with $U_\alpha U_\beta$ and by taking into account (24)_{2,3} we obtain

$$(g - \lambda + \lambda^E)^{(1)} = 0, \quad U^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(1)} = 0 \tag{25}$$

By contracting Equation (24)₂ with $U_\alpha h_\beta^\delta$ and by taking into account (24)₃ we obtain

$$(g_\delta - \lambda_\delta + \lambda_\delta^E)^{(1)} = -\frac{3}{\rho c^4 \theta_{1,2}} \frac{k_B}{m} q_\delta \tag{26}$$

There remains to contract Equation (24)₁ with h_α^δ and Equation (24)₂ with $h_\alpha^\delta h_\beta^\psi$; by using Equations (28), (26) the result is

$$\left(q_L^\delta\right)^{(1)} = -\frac{3p}{\rho c^4 \theta_{1,2}} q^\delta, \quad \left(P^{\delta\psi}\right)^{(1)} = \pi h^{\delta\psi} + t^{<\delta\psi>} \tag{27}$$

Regarding the first term of this equation we note that it can be rewritten by using the recurrence relations

found by Arima, Carrisi, Pennisi and Ruggeri in^[13], Equation (16) and Equation (12)_{1,2}; it becomes

$$\left(q_L^\delta\right)^{(1)} = -\frac{\rho}{e+p} q^\delta$$

which is the same value found by Carrisi and Pennisi in (7)₁ of^[3] which concerned an approach with a linear deviation from equilibrium. Since we will see in the next subsection that $\left(q_L^\delta\right)^{(2)} \neq 0$, the expression that Carrisi and Pennisi found in Equation (7)₁ of^[3] cannot be assumed to hold up to whatever order with respect to equilibrium.

4.2. The expressions of q_L^α and $P^{\alpha\beta}$ at second order with respect to equilibrium

We have to consider Equation (23)₂; to this end, we need

$$\left(e^{-\frac{1}{k_B} \left[m(g-\lambda+\lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right]} - 1\right)^{(2)}$$

and

$$\left(e^{-\frac{1}{k_B} \left[m(g-\lambda+\lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right]} - 1\right)^{(1)} \tag{28}$$

Since the Taylor's expansion of the function e^x around $x = 0$ and up to second order is $1 + x + \frac{x^2}{2}$, we have

$$\begin{aligned} &\left(e^{-\frac{1}{k_B} \left[m(g-\lambda+\lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right]} - 1\right)^{(2)} = -\frac{m}{k_B} (g - \lambda + \lambda^E)^{(2)} - \\ &\frac{1}{k_B} \left[(g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right] + \frac{1}{2} \left(\frac{m (g - \lambda + \lambda^E)^{(1)} }{k_B} \right)^2 + \\ &+ \frac{1}{(k_B)^2} \left[m (g - \lambda + \lambda^E)^{(1)} (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(1)} p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right] \\ &+ \frac{1}{2 (k_B)^2} \left[(g_\mu - \lambda_\mu + \lambda_\mu^E)^{(1)} p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right] \left[(g_\nu - \lambda_\nu + \lambda_\nu^E)^{(1)} p^\nu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right], \end{aligned}$$

$$\begin{aligned} &\left(e^{-\frac{1}{k_B} \left[m(g-\lambda+\lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right]} - 1\right)^{(1)} = -\frac{m}{k_B} (g - \lambda + \lambda^E)^{(1)} - \\ &\frac{1}{k_B} \left[(g_\mu - \lambda_\mu + \lambda_\mu^E)^{(1)} p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right] \end{aligned}$$

By using (28) and (26), we obtain

$$\begin{aligned} &\left(e^{-\frac{1}{k_B} \left[m(g-\lambda+\lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right]} - 1\right)^{(2)} = -\frac{m}{k_B} (g - \lambda + \lambda^E)^{(2)} - \\ &\frac{1}{k_B} \left[(g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} p^\mu \left(1 + \frac{\mathcal{I}}{m c^2}\right) \right] + \frac{1}{2} \left(\frac{3}{\rho c^4 \theta_{1,2}} \frac{1}{m} \right)^2 (q_\mu p^\mu)^2 \left(1 + \frac{\mathcal{I}}{m c^2}\right)^2, \end{aligned}$$

$$\left(e^{-\frac{1}{k_B} \left[m(g - \lambda + \lambda^E) + (g_\mu - \lambda_\mu + \lambda_\mu^E) p^\mu \left(1 + \frac{\mathcal{I}}{m c^2} \right) \right]} - 1 \right)^{(1)} = \frac{3}{\rho c^4 \theta_{1,2}} \frac{1}{m} q_\mu p^\mu \left(1 + \frac{\mathcal{I}}{m c^2} \right)$$

By using these expressions in (23)₂ it becomes

$$\begin{aligned} \left(\frac{f^L - f_E}{f_E} \right)^{(2)} &= -\frac{m}{k_B} (g - \lambda + \lambda^E)^{(2)} - \\ \frac{1}{k_B} \left[(g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} p^\mu \left(1 + \frac{\mathcal{I}}{m c^2} \right) \right] &+ \frac{1}{2} \left(\frac{3}{\rho c^4 \theta_{1,2}} \frac{1}{m} \right)^2 (q_\mu p^\mu)^2 \left(1 + \frac{\mathcal{I}}{m c^2} \right)^2 + \\ + \left(\frac{f^{Ek}}{f_E} \right)^{(1)} \frac{3}{\rho c^4 \theta_{1,2}} \frac{1}{m} q_\mu p^\mu \left(1 + \frac{\mathcal{I}}{m c^2} \right) &+ \left(\frac{f^{Ek}}{f_E} - 1 \right)^{(2)} \end{aligned}$$

We can now substitute this result in (22) so that the homogeneous second order part of this system is

$$\begin{aligned} (q_L^\alpha)^{(2)} &= m c \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \left(\frac{f^L - f_E}{f_E} \right)^{(2)} p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = \\ &= -\frac{m}{k_B} (g - \lambda + \lambda^E)^{(2)} \rho U^\alpha - \frac{m}{k_B} (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} T_E^{\alpha\mu} + \\ &+ \frac{1}{2} \left(\frac{3}{\rho c^4 \theta_{1,2}} \right)^2 A_E^{\alpha\mu\nu} q_\mu q_\nu + \frac{3}{\rho c^4 \theta_{1,2}} q_\mu \left(T_{Ek}^{\alpha\mu} \right)^{(1)} + \underline{(V_{Ek}^\alpha)^{(2)}} = \\ &= -\frac{m}{k_B} (g - \lambda + \lambda^E)^{(2)} \rho U^\alpha - \frac{m}{k_B} (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} T_E^{\alpha\mu} + \\ &+ \frac{3}{\rho c^4 \theta_{1,2}} q_\mu \left(\pi h^{\alpha\mu} + \frac{1}{2 c^2} U^\alpha q^\mu + t^{<\alpha\mu>} \right), \end{aligned} \tag{29}$$

$$\begin{aligned} (P^{\alpha\beta})^{(2)} &= c \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \frac{f^L - f_E}{f_E} p^\alpha p^\beta \left(1 + \frac{\mathcal{I}}{m c^2} \right) p^\mu \left(1 + \frac{\mathcal{I}}{m c^2} \right) \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = \\ &= -\frac{m}{k_B} (g - \lambda + \lambda^E)^{(2)} T_E^{\alpha\beta} - \frac{m}{k_B} (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} A_E^{\alpha\beta\mu} + \\ &+ \frac{1}{2} \left(\frac{3}{\rho c^4 \theta_{1,2}} \right)^2 A_E^{\alpha\beta\mu\nu} q_\mu q_\nu + \frac{3}{\rho c^4 \theta_{1,2}} q_\mu \left(A_{Ek}^{\alpha\beta\mu} \right)^{(1)} + \underline{(T_{Ek}^{\alpha\beta})^{(2)}}, \\ U_\alpha (q_L^\alpha)^{(2)} &= 0, \quad U_\alpha (P^{\alpha\beta})^{(2)} = 0 \end{aligned}$$

where the underlined terms are zero, while

$$\begin{aligned} A_E^{\alpha\beta\mu\nu} &= \rho \theta_{0,3} U^\alpha U^\beta U^\mu U^\nu + \rho c^2 \theta_{1,3} U^{(\alpha} U^\beta h^{\mu\nu)} + \rho c^4 \theta_{2,3} h^{(\alpha\beta} h^{\mu\nu)}, \\ (A_{Ek}^{\alpha\beta\mu})^{(1)} &= \frac{1}{4c^4} \Delta U^\alpha U^\beta U^\mu - \left(\frac{3}{4c^2} \frac{N^\Delta}{D_4} \Delta + 3 \frac{N^\Pi}{D_4} \Pi \right) h^{(\alpha\beta} U^\mu) \\ &+ \frac{3}{c^2} \frac{N_3}{D_3} q^{(\alpha} U^\beta U^\mu) + \frac{3}{5} \frac{N_{31}}{D_3} h^{(\alpha\beta} q^\mu) + 3C_5 t^{<\alpha\beta>3} U^\mu \end{aligned}$$

as it can be seen from Equations (14) and (35) found by Arima, Carrisi, Pennisi and Ruggeri in^[13] (Δ is the 15th variable). By contracting Equation (29)₁ with $\frac{U_\alpha}{c^2}$, Equation (29)₂ with $\frac{U_\alpha U_\beta}{c^4}$ and by taking into account

(29)_{3,4} we obtain

$$\begin{aligned}
 0 &= -\frac{m}{k_B} \rho (g - \lambda + \lambda^E)^{(2)} - \frac{m}{k_B} \frac{e}{c^2} U^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} + \frac{3}{2 \rho c^6 \theta_{1,2}} q_\mu q^\mu, \\
 0 &= -\frac{m}{k_B} \frac{e}{c^2} (g - \lambda + \lambda^E)^{(2)} - \frac{m}{k_B} \rho \theta_{0,2} U^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} + \\
 &+ \left(-\frac{3}{4} \frac{\theta_{1,3}}{\rho c^6 (\theta_{1,2})^2} + \frac{3}{\rho c^6 \theta_{1,2}} \frac{N_3}{D_3} \right) q^\mu q_\mu
 \end{aligned}$$

This system fully determine $(g - \lambda + \lambda^E)^{(2)}$ and $U^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)}$.

By contracting Equation (29)₂ with $U_\alpha h_\beta^\delta$ and by taking into account (29)_{3,4} we obtain

$$\begin{aligned}
 h^{\mu\delta} (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} &= -\frac{k_B}{m} \frac{9}{\rho^2 c^8 (\theta_{1,2})^2} q_\mu \left(A_{Ek}^{\alpha\beta\mu} \right)^{(1)} U_\alpha h_\beta^\delta = \\
 &= \frac{k_B}{m} \frac{9}{\rho^2 c^8 (\theta_{1,2})^2} \left[\left(\frac{1}{4} \frac{N^\Delta}{D_4} \Delta + c^2 \frac{N^\Pi}{D_4} \Pi \right) q^\delta + C_5 c^2 t^{<\delta\mu>} q_\mu \right]
 \end{aligned} \tag{30}$$

There remains to contract Equation (29)₁ with h_α^δ and Equation (29)₂ with $h_\alpha^\delta h_\beta^\psi$; the result is

$$\begin{aligned}
 \left(q_L^\delta \right)^{(2)} &= -\frac{m p}{k_B} h^{\delta\mu} (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} + \frac{3}{\rho c^4 \theta_{1,2}} \left(-\pi q^\delta + t^{<\delta\mu>} q_\mu \right), \\
 \left(P^{\delta\psi} \right)^{(2)} &= -\frac{m}{k_B} \left[p (g - \lambda + \lambda^E)^{(2)} + \frac{1}{3} \rho c^2 \theta_{1,2} U^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)} \right] h^{\delta\psi} + \\
 &+ \frac{3}{\rho c^4 \theta_{1,2}} \left(\frac{1}{2} \frac{\theta_{2,3}}{\theta_{1,2}} - \frac{1}{5} \frac{N_{31}}{D_3} \right) \left(2 q^\delta q^\psi - q^\mu q_\mu h^{\delta\psi} \right)
 \end{aligned} \tag{31}$$

Obviously, here the above found expressions of $(g - \lambda + \lambda^E)^{(2)}$, $U^\mu (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)}$ and of $h^{\delta\mu} (g_\mu - \lambda_\mu + \lambda_\mu^E)^{(2)}$ must be used.

5. Conclusions

We have obtained the gradient form of the balance equations not only for their left hand sides, but also for their right hand sides; this result has been obtained both with the Eckart approach and the Landau-Lifshitz one. In this way the symmetric form and a 4-potential is also obtained for both sides. Moreover, this form allowed to prove the hyperbolicity of the equations and the H-theorem up to whatever order. We have also seen that the Eckart approach and the Landau-Lifshitz one are equivalent if the Lagrange multipliers $\lambda_{\alpha_1 \dots \alpha_n}$ are taken as independent variables. The difference comes only after a different definition of the deviations of $\lambda - \lambda^E$ and of $\lambda_\alpha - \lambda_\alpha^E$; in any case, there is an invertible transformation between their independent variables.

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Conflict of interest

The author declares no conflict of interest.

References

1. Eckart C. The Thermodynamics of Irreversible Processes. III. Relativistic Theory of the Simple Fluid. *Physical Review*. 1940, 58(10): 919-924. doi: <https://doi.org/10.1103/physrev.58.919>
2. Landau LD, Lifshitz EM, et.al. Butterworth-Heinemann, Oxford, 1997.
3. Carrisi MC, Pennisi S. Relativistic extended thermodynamics of polyatomic gases in the Landau and Lifshitz description. *International Journal of Non-Linear Mechanics*. 2021, 135: 103756. doi: <https://doi.org/10.1016/j.ijnonlinmec.2021.103756>
4. Carrisi MC, Pennisi S. Hyperbolicity of a model for polyatomic gases in relativistic extended thermodynamics. *Continuum Mechanics and Thermodynamics*. 2019, 32(5): 1435-1454. doi: <https://doi.org/10.1007/s00161-019-00857-0>
5. Pennisi S. Consistent Order Approximations in Extended Thermodynamics of Polyatomic Gases. *Journal of Nature, Science & Technology*. 2021, 1(2): 12-21. doi: <https://doi.org/10.36937/janset.2021.002.003>
6. Brini F. Hyperbolicity region in extended thermodynamics with 14 moments. *Continuum Mechanics and Thermodynamics*. 2001, 13(1): 1-8. doi: <https://doi.org/10.1007/s001610100036>
7. Ruggeri T, Trovato M. Hyperbolicity in Extended Thermodynamics of Fermi and Bose gases. *Continuum Mechanics and Thermodynamics*. 2004, 16(6): 551-576. doi: <https://doi.org/10.1007/s00161-004-0180-6>
8. Brini F, Ruggeri T. Second-order approximation of extended thermodynamics of a monatomic gas and hyperbolicity region. *Continuum Mechanics and Thermodynamics*. 2019, 32(1): 23-39. doi: <https://doi.org/10.1007/s00161-019-00778-y>
9. Struchtrup H. Heat transfer in the transition regime: Solution of boundary value problems for Grad's moment equations via kinetic schemes. *Physical Review E*. 2002, 65(4). doi: <https://doi.org/10.1103/physreve.65.041204>
10. Struchtrup H. Stable transport equations for rarefied gases at high orders in the Knudsen number. *Physics of Fluids*. 2004, 16(11): 3921-3934. doi: <https://doi.org/10.1063/1.1782751>
11. Demontis F, Pennisi S, On two possible ways to recover Ordinary Thermodynamics from Extended Thermodynamics of Polyatomic gases, accepted for publication in *Journal of Physics A: Mathematical and Theoretical* (2023).
12. Pennisi S, Ruggeri T. Relativistic extended thermodynamics of rarefied polyatomic gas. *Annals of Physics*. 2017, 377: 414-445. doi: <https://doi.org/10.1016/j.aop.2016.12.012>
13. Arima T, Carrisi MC, Pennisi S, et al. Relativistic Rational Extended Thermodynamics of Polyatomic Gases with a New Hierarchy of Moments. *Entropy*. 2021, 24(1): 43. doi: <https://doi.org/10.3390/e24010043>
14. Pennisi S, Ruggeri T. A New BGK Model for Relativistic Kinetic Theory of Monatomic and Polyatomic Gases. *Journal of Physics: Conference Series*. 2018, 1035: 012005. doi: <https://doi.org/10.1088/1742-6596/1035/1/012005>
15. De Groot SR, van Leeuwen WA, et.al. *Relativistic Kinetic Theory. Principles and Applications*, North Holland, Amsterdam, New York, Oxford, 1980.
16. Rezzolla L, Zanotti O. *Relativistic Hydrodynamics*, Oxford University Press, Oxford, 2013.

Appendix: Proof of the unicity of the solution of Equations (10) and of $g^E = 0$, $g_\mu^E = 0$

The equations (10) calculated at equilibrium are identically satisfied; so they are equivalent to their derivatives with respect to $\lambda_{\alpha_1 \dots \alpha_{n_1}}$ (with $n_1 \geq 2$) which now we express, by using a compact notation, as $\lambda_{A_{n_1}}$. These derivatives are

$$\begin{aligned} & \frac{U_\alpha}{-k_B} \int_{\mathbb{R}^3} \int_0^{+\infty} f_E F \left[m \frac{\partial g}{\partial \lambda_{A_{n_1}}} + p^\mu \frac{\partial g_\mu}{\partial \lambda_{A_{n_1}}} \left(1 + \frac{\mathcal{I}}{m c^2} \right) + \frac{\partial \Delta \chi}{\partial \lambda_{A_{n_1}}} \right] p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = 0 \\ & \frac{U_\alpha}{-k_B} \int_{\mathbb{R}^3} \int_0^{+\infty} f_E F \left[m \frac{\partial g}{\partial \lambda_{A_{n_1}}} + p^\mu \frac{\partial g_\mu}{\partial \lambda_{A_{n_1}}} \left(1 + \frac{\mathcal{I}}{m c^2} \right) + \frac{\partial \Delta \chi}{\partial \lambda_{A_{n_1}}} \right] p^\alpha p^\beta \left(1 + \frac{\mathcal{I}}{m c^2} \right) \cdot \\ & \quad \cdot \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = 0, \quad \text{with} \quad F = e^{\frac{-1}{k_B} \left[m g + p^\mu g_\mu \left(1 + \frac{\mathcal{I}}{m c^2} \right) + \Delta \chi \right]} \end{aligned} \quad (32)$$

which, calculated at equilibrium, give

$$\begin{aligned} & - \frac{m}{k_B} \left[\left(\frac{\partial g}{\partial \lambda_{A_{n_1}}} \right)_E U_\alpha V_E^\alpha + \left(\frac{\partial g_\mu}{\partial \lambda_{A_{n_1}}} \right)_E U_\alpha T_E^{\alpha\mu} \right] = \frac{m}{k_B} U_\alpha A_E^{\alpha A_{n_1}} \\ & - \frac{m}{k_B} \left[\left(\frac{\partial g}{\partial \lambda_{A_{n_1}}} \right)_E U_\alpha T_E^{\alpha\beta} + \left(\frac{\partial g_\mu}{\partial \lambda_{A_{n_1}}} \right)_E U_\alpha A_E^{\alpha\beta\mu} \right] = \frac{m}{k_B} U_\alpha A_E^{\alpha\beta A_{n_1}} \end{aligned}$$

By contracting these equations with $\lambda_{A_{n_1}}$ and taking the sum for $n_1 = 2, \dots, N$ they become

$$\begin{aligned} g^{(1)} U_\alpha V_E^\alpha + g_\mu^{(1)} U_\alpha T_E^{\alpha\mu} &= - \sum_{n_1=2}^N U_\alpha A_E^{\alpha A_{n_1}} \lambda_{A_{n_1}}, \\ g^{(1)} U_\alpha T_E^{\alpha\beta} + g_\mu^{(1)} U_\alpha A_E^{\alpha\beta\mu} &= - \sum_{n_1=2}^N U_\alpha A_E^{\alpha\beta A_{n_1}} \lambda_{A_{n_1}} \end{aligned} \quad (33)$$

The first one of these equations, the second one contracted with U_β and the second one contracted with h_β^δ constitute a system by using also the expressions

$$V_E^\alpha = \rho U^\alpha, \quad T_E^{\alpha\mu} = \frac{e}{c^2} U^\alpha U^\mu + p h^{\alpha\mu}, \quad A_E^{\alpha\beta\mu} = \rho \theta_{0,2} U^\alpha U^\beta U^\mu + \rho c^2 \theta_{1,2} U^{(\alpha} h^{\beta\mu)}$$

This system has the unique solution

$$g^{(1)} = \begin{vmatrix} \rho & \frac{e}{c^2} \\ \frac{e}{c^2} & \rho \theta_{0,2} \end{vmatrix}^{-1} \sum_{n_1=2}^N \left(\frac{e}{c^6} U_\alpha U_\beta A_E^{\alpha\beta A_{n_1}} - \rho \theta_{0,2} U_\alpha A_E^{\alpha A_{n_1}} \right) \lambda_{A_{n_1}}$$

$$U^\mu g_\mu^{(1)} = \begin{vmatrix} \rho & \frac{e}{c^2} \\ \frac{e}{c^2} & \rho \theta_{0,2} \end{vmatrix}^{-1} \sum_{n_1=2}^N \left(\frac{e}{c^4} U_\alpha A_E^{\alpha A_{n_1}} - \frac{\rho}{c^4} U_\alpha U_\beta A_E^{\alpha\beta A_{n_1}} \right) \lambda_{A_{n_1}}$$

$$h^{\mu\delta} g_\mu^{(1)} = \frac{3}{\rho c^4 \theta_{1,2}} U_\alpha h_\beta^\delta \sum_{n_1=2}^N A_E^{\alpha\beta A_{n_1}} \lambda_{A_{n_1}}$$

which fully determine $g^{(1)}$ and $g_\mu^{(1)}$.

If we want their second order homogeneous parts, we have to take the derivatives of (32) with respect to $\lambda_{A_{n_2}}$. These derivatives are equivalent to

$$\begin{aligned} & \frac{U_\alpha}{-k_B} \int_{\mathbb{R}^3} \int_0^{+\infty} f_E F \left[m \frac{\partial^2 g}{\partial \lambda_{A_{n_2}} \partial \lambda_{A_{n_1}}} + p^\mu \frac{\partial^2 g_\mu}{\partial \lambda_{A_{n_2}} \partial \lambda_{A_{n_1}}} \left(1 + \frac{\mathcal{I}}{m c^2} \right) + \frac{\partial^2 \Delta \chi}{\partial \lambda_{A_{n_2}} \partial \lambda_{A_{n_1}}} \right] \\ & \cdot p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = -U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \frac{\partial F}{\partial \lambda_{A_{n_2}}} \frac{\partial F}{\partial \lambda_{A_{n_1}}} p^\alpha \varphi(\mathcal{I}) d\mathcal{I} d\vec{P}, \\ & \frac{U_\alpha}{-k_B} \int_{\mathbb{R}^3} \int_0^{+\infty} f_E F \left[m \frac{\partial^2 g}{\partial \lambda_{A_{n_2}} \partial \lambda_{A_{n_1}}} + p^\mu \frac{\partial^2 g_\mu}{\partial \lambda_{A_{n_2}} \partial \lambda_{A_{n_1}}} \left(1 + \frac{\mathcal{I}}{m c^2} \right) + \frac{\partial^2 \Delta \chi}{\partial \lambda_{A_{n_2}} \partial \lambda_{A_{n_1}}} \right] \\ & \cdot p^\alpha p^\beta \left(1 + \frac{\mathcal{I}}{m c^2} \right) \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} = \\ & = -U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} f_E \frac{\partial F}{\partial \lambda_{A_{n_2}}} \frac{\partial F}{\partial \lambda_{A_{n_1}}} p^\alpha p^\beta \left(1 + \frac{\mathcal{I}}{m c^2} \right) \varphi(\mathcal{I}) d\mathcal{I} d\vec{P} \end{aligned}$$

If we calculate these equations at equilibrium, contract the result with $\frac{1}{2} \lambda_{A_{n_2}} \lambda_{A_{n_1}}$ and take the sum for $n_1, n_2 = 2, \dots, N$ they give a system like (33), but with $g^{(2)}$ and $g_\mu^{(2)}$ instead of $g^{(1)}$ and $g_\mu^{(1)}$ in the left hand sides, while in the right hand sides there are known functions which include polynomials in $g^{(1)}$ and $g_\mu^{(1)}$ (already known). Therefore, by proceeding as we have done for (33), we fully determine $g^{(2)}$ and $g_\mu^{(2)}$.

It is obvious that, by taking the higher order derivatives of (32) and proceeding in the same way, we obtain $g^{(h)}$ and $g_\mu^{(h)} \forall h$. This completes the proof that (10) jointly with $g^E = 0, g_\mu^E = 0$ give one and only one solution.