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Modified Hermite Polynomials for Solving Quadratic Optimal Control Problems

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Abstract: The main purpose of this paper is the construction an explicit formula for modified Hermit function differentiation operational matrix and other new properties. Then an efficient approximate method is investigated for treating quadratic optimal control problem with the aid of the derived operation matrix. The technique essentially based on reducing the optimal problem indirectly to a system of linear algebraic equations in the expansion of unknown coefficients. The obtained numerical results are compared with the exact one.

Keywords: Hermit function, quadratic optimal control problem, differentiation operational matrix
Received: 15th Jul. 2019 **Accepted:** 15th Aug. 2019 **Online:** 23rd Aug. 2019

1. Introduction

Spectral methods based on state vector parameterization are one of the discritirazation methods for approximate solution of differential equations, integral equations, and optimal control problem[1-10]. The subject of optimal control problems is an important part of mathematics. Many practical computing techniques have been presented in optimal control[11-18].

The current technique is based upon the expansion of state variable in modified Hermite functions having unknown coefficients. The proposed algorithm reduces the quadratic optimal control problems into a system of algebraic equations using indirect method.

The organization of the paper is: in the next section, the definition of modified Hermite polynomials is introduced and the discussion about the basic conversion from power form to modified Hermite polynomials is included. In section 3, the computational for operational matrix of derivative is listed for modified Hermite polynomials. An algorithm for treating quadratic optimal control problem is proposed in section 4 depending on differentiation operational matrix of derivative. Section 5 contains a test example while the conclusion is presented in the last section followed by the description of the numerical example.

2. Modified Hermite Polynomials

Modified Hermite polynomials $MH_n(t)$ are important polynomials, defined as follows

$$MH_n(t) = 2^{-\frac{n}{2}} H_n\left(\frac{t}{\sqrt{2}}\right) \quad (1)$$

where $H_n(t)$ are n^th Hermite polynomials, and can be calculated by the following recurrence relation[14,15]

$$H_{n+1}(t) = 2t\, H_n(t) - 2nH_{n-1}(t) \quad (2)$$

3. New Differentiation Operational Matrix for $MH_n(t)$

A new property of modified Hermite functions has been derived in this section. The few modified Hermite polynomials can be obtained using Eq. 1

$MH_1(t) = t$
$MH_2(t) = t^2 - 1$
$MH_3(t) = t^3 - 3t$
$MH_4(t) = t^4 - 6t^2 + 3$
$MH_5(t) = t^5 - 10t^3 + 15t$ (3)

Lemma (1)

The polynomials $MH_n(t)$ can be defined from the following recurrence relation

$$MH_{n+1}(t) = t\, MH_n(t) - nMH_{n-1}(t)$$

where $MH_0(t) = 1$ and $MH_1(t) = t$

the derivative of $MH_n(t)$ are,

$\dot{M}H_0(t) = 0$

$\dot{M}H_1(t) = 1$

$\dot{M}H_2(t) = 2t$

$\dot{M}H_3(t) = 3t^2 - 3$

$\dot{M}H_4(t) = 4t^3 - 12\mathrm{t}$

$\dot{M}H_5(t) = 5t^4 - 30t^2 + 15$ (4)

New, the derivative $\dot{M}H_n(t)$ can be written in terms of $MH_n(t)$,

$\dot{M}H_0(t) = 0$

$\dot{M}H_1(t) = MH_0(t)$

$\dot{M}H_2(t) = 2MH_1(t)$

$\dot{M}H_3(t) = 3MH_2(t)$

$\dot{M}H_4(t) = 4MH_3(t)$

$\dot{M}H_5(t) = 5MH_4(t)$

In general

$$\dot{M}H_n(t) = nMH_{n-1}(t) \quad (5)$$

As a result the differentiation operational matrix for $MH_n(t)$ is given by

$$D_{MH} = \begin{pmatrix} 0 & d_{22} & 0 & \dots & 0 \\ 0 & 0 & d_{33} & \dots & 0 \\ 0 & 0 & 0 & d_{44} & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & d_{nn} \end{pmatrix}$$

where the elements of the first row are zero,

$$d_{ij} = \begin{cases} i & i = j \\ 0 & otherwise \end{cases} \quad (6)$$

1. The Proposed Algorithm

The spectral method with state vector parameterization using modified Hermite polynomials as basic functions are used to solve quadratic optimal control problem,

minimize $J = \int_a^b (x^T Q x + u^T R u) dt$
(7)

subject to $\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{Bu}$ (8)

and $x(a) = \alpha$ (9)

where $A \in R^n \times R^n, B \in R^n \times R^m, x \in R^n, u \in R^m$, the matrix Q is $n \times n$ positive semi definite matrix $(x^T Q x \geq 0)$ and R is $m \times m$ positive definite matrix, i.e., $u^T R u > 0$ unless $u(t) = 0$.

The algorithm is designed for solving Eqs. 7-9 and can be summarized by the following steps:

Step1: Find the necessary condition for optimality.

$$\dot{x} = Ax - \frac{1}{2} B R^{-1} B^T \lambda \quad (10)$$

$$\dot{\lambda} = -\,2Qx - A^T \lambda \quad (11)$$

$$u = -\frac{1}{2} R^{-1} B^T \lambda \quad (12)$$

With the boundary conditions

$$x(a) = \alpha \quad (13)$$

$$\lambda(b) = \beta \quad (14)$$

Step 2: Approximate $x(t)$ and $\lambda(t)$ by finite length modified Hermite polynomials,

$$x = \delta_1 MH, \lambda = \delta_2 MH$$

Step 3: Construct the square system of algebraic equations

$\dot{x} = \delta_1 D_{MH} MH$ (15)

$\dot{\lambda} = \delta_2 D_{MH} MH$ (16)

By substituting Eqs. 15-16 into Eqs. 10-12, one can obtain

$$\delta_1 D_{MH} MH = \left(A\delta_1 - \frac{1}{2} BR^{-1} B^T \delta_2\right) MH$$

$$\delta_2 D_{MH} MH = (-\,2Q\delta_1 - A^T \delta_2) MH$$

or

$\delta_1 D_{MH} MH = T_1 MH$ (17)

$\delta_2 D_{MH} MH = T_2 MH$ (18)

where $T_1 = A\delta_1 - \frac{1}{2} BR^{-1} B^T \delta_2$

and $T_2 =- 2Q\delta_1 - A^T \delta_2$

step 4: Approximate Equn. (13-14) using modified Hermite polynomials, yields

$x(a) = \delta_1 MH(a) = \alpha$ (19)

and $\lambda(b) = \delta_2 MH(b) = \beta$ (20)

step 5: Solve the system of algebraic Eqs. 17 to 20 to find the entries of the two vectors δ_1 and δ_2.

Step 6: Calculate the control variable from Eq 12.

Step 7: Evaluate the cost function

$$J = \int_a^b [(MH)^T V_1 (MH) + (MH)^T V_2 (MH)] dt$$

2. Numerical Example:

minimize $J = \int_0^1 u^2(t) dt$

with the constrains

$\dot{x_1} = x_2 \quad x_1(0) = 1 \quad x_1(1) = 1$

$\dot{x_2} = u \quad x_2(0) = 1 \quad x_2(1)$ free

The exact solution is

$$x_1(t) = t^3 - 3t^2 + t + 1$$

$$x_2(t) = 3t^2 - 6t + 1$$

$$u(t) = 6(t-1) \quad 0 \le t \le 1$$

and the value of the performance index is $J = 12$.

Step 1:

The necessary conditions for optimality will be:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\frac{1}{2}\lambda_2$$

$$\dot{\lambda_1} = 0$$

$$\dot{\lambda_2} =- \lambda_1$$

with the boundary conditions

$x_1(0) = 1$, $x_1(1) = 1,\ x_2(0) = 1 \quad \lambda_2(1) = 0$

Step 2: approximate $x_1(t)$ and $\lambda_2(1)$ using modified Hermite polynomials with $n = 3$

$$x_1(t) = a_0 MH_0(t) + a_1 MH_1(t) + a_2 MH_2(t) + a_3 MH_3(t)$$

$$\lambda_2(t) = b_0 MH_0(t) + b_1 MH_1(t) + b_2 MH_2(t) + b_3 MH_3(t)$$

or

$x_1 = \delta_1 MH$ and $\lambda_2(t) = \delta_2 MH$

where

$$\delta_1 = (a_0 \quad a_1 \quad a_2 \quad a_3)$$

$$\delta_2 = (b_0 \quad b_1 \quad b_2 \quad b_3)$$

and

$$MH = (MH_0 \quad MH_1 \quad MH_2 \quad MH_3)^T$$

Now $x_2(t)$ and $\lambda_1(t)$ can be found as follows:

$$x_2 = \dot{x_1} = \delta_1 D_{MH} MH$$

$$\lambda_1 = -\dot{\lambda_2} = \delta_2 D_{MH} MH$$

where

$$D_{MH} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

Step 3: Approximate the boundary conditions

$$x_1(0) = 1 \Longrightarrow a_0 MH_0(0) + a_1 MH_1(0) + a_2 MH_2(0) + a_3 MH_3(0) = 1$$

Therefore,

$a_0 - a_2 = 1$

$x_1(1) = 1 \Rightarrow a_0 + a_1 - 2a_3 = 1$

$x_2(0) = 1 \Rightarrow a_1 - 3a_3 = 1$

$\lambda_2(1) = 0 \Rightarrow b_0 + b_1 - 2b_3 = 0$

Finally, the approximated results will be

$x_1(t) = H_3 - 3H_2 + 4H_1 - 2H_0$

$x_2(t) = 3H_2 - 6H_1 + 4H_0$

$u(t) = 6H_1 - 6H_0$

and $J = 12$ which is equal to the exact one.

4. Conclusion

In this work, a novel algorithm for solving quadratic optimal control problem is suggested. The presentation of the algorithm is essentially based on the proposed operational matrix of derivative for modified Hermite polynomials. The differentiation operational matrix is used to convert the original problem into algebraic equations and has the following advantages; first; all its elements are integers and hence there is no truncation error. Second, easy to construct because it has special structure.

References

1. S. N. Al-Rawi, Numerical Solution of Integral Equa tions Using Taylor Series, Journal of the College of Education, 5 (1992) 51-60.
2. M. N. Sahlan, H. Feyzollahzadeh, Operational matri ces of Chebyshev polynomials for solving singular Volterra integral equations, Math Sci. 11 (2017) 165–171.
3. F. Dkhilalli, S. M. Borchani, M. Rasheed, R. Barille, S. Shihab, K. Guidara, M Megdiche, Characterizations and morphology of sodium tungstate particles, Royal Society open science, 5 (8) (2018) 1-16.
4. J. A. Eleiwy, S. N. Shihab, Chebyshev Polynomials and Spectral Method for Optimal Control Problem, Engineering and Technology Journal, 27 (14) (2009) 2642-2652.
5. E. H. Ouda, An Approximate Solution of some Varia tional Problems Using Boubaker Polynomials, Baghdad Science Journal, 5 (1) (2018) 106-109.
6. S. N. SHIHAB, M. A. Sarhan, New Operational Ma trices of Shifted Fourth Chebyshev wavelets, Elixir International Journal-Applied Mathematics, 69 (1) (2014) 23239-23244.
7. S. N. Shihab, T. N. Naif, On the orthonormal Bern stein polynomial of order eight, open Science Journal of Mathematics and Application, 2 (2) (2014) 15-19.
8. S. N. Shihab, M. A. Sarhan, Convergence analysis of shifted fourth kind Chebyshev wavelets, IOSR journal of mathematics, 10 (2) (2014) 54-58.
9. Z. Yang, H. Zhang, Chebyshev polynomials for ap proximation of solution of fractional partial differential equations with variable coefficients, in: Atlantis Press, (IC3ME 2015) (2015) 252-260.
10. S. Al-Rawi, On the solution of Certain Fractional In tegral Equations, Kirkuk University Journal for Scientific Studies 1 (2), (2006) 125-136.
11. S. Al-Rawi, NUMERICAL SOLUTION OF INTE GRAL EQUATIONS USING TAYLOR SERIES, Journal of the College of Education 5, (1998) 51-60.
12. M. Delphi, S. SHIHAB, Operational Matrix Basic Spline Wavelets of Derivative for linear Optimal Control Problem, Electronics Science Technology and Application 6 (2), (2019) 18-24.
13. S. N. Al-Rawi, F. A. Al-Heety, S. S. Hasan, A New Computational Method for Optimal Control Problem with B-spline Polynomials, Engineering and Technology Journal, 28 (18) (2010) 5711-5718.
14. S. N. Al-Rawi, H. R. Al-Rubaie, An Approximate solution of some continuous time Linear-Quadratic optimal control problem via Generalized Laguerre Polynomial, Journal of Pure and Applied Sciences, 22 (1) (2010) 85-97.
15. I. N. Ahmed Fahmi, S. F. Ibraheem, E. H. Ouda, In direct Method for Optimal Control Problem Using Boubaker Polynomial, Baghdad Science Journal, 13 (1) (2016) 183-189.
16. E. H. Ouda, A new Approach for Solving Optimal Control Problems Using Normalized Boubaker Polynomials, Emirates Journal for Engineering Research, 23 (1) (2018) 1-12.
17. A. Wünsche, Associated Hermite Polynomials Re lated to Parabolic Cylinder Functions, Scientific Research, 9 (1) (2019).
18. C. Cesarano, Operational Methods and New Identi ties for Hermite Polynomials, Math. Model. Nat. Phenom. 12 (3) (2017) 44-50.